ON CONFORMALLY COVARIANT POWERS OF THE LAPLACIAN

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Abstract. We propose and discuss recursive formulas for conformally covariant powers P_{2N} of the Laplacian (GJMS-operators). For locally conformally flat metrics, these describe the non-constant part of any GJMS-operator as the sum of a certain linear combination of compositions of lower order GJMS-operators (primary part) and a second-order operator which is defined by the Schouten tensor (secondary part). We complete the description of GJMS-operators by proposing and discussing recursive formulas for their constant terms, i.e., for Branson's Q-curvatures, along similar lines. We confirm the picture in a number of cases. Full proofs are given for spheres of any dimension and arbitrary signature. Moreover, we prove formulas of the respective critical third power P_6 in terms of the Yamabe operator P_2 and the Paneitz operator P_4, and of a fourth power in terms of P_2, P_4 and P_6. For general metrics, the latter involves the first two of Graham's extended obstruction tensors [G4]. In full generality, the recursive formulas remain conjectural. We describe their relation to the theory of residue families and the associated Q-polynomials as developed in [J1].

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1. Introduction

The Laplace-Beltrami operator Δ_g of a Riemannian manifold (M,g) is one of the basic geometric differential operators. Its significance rests on its invariance with respect to isometries. In two dimensions, it is also invariant (or rather covariant) with respect to conformal changes $g \mapsto e^{2\varphi}g$ of the metric. Although this is not true in dimension $n \geq 3$, the operator

$$P_2(g) = \Delta_g - \left(\frac{n}{2}-1\right)\mathsf{J}_g, \quad \mathsf{J}_g = \tau_g/2(n-1), \tag{1.1}$$

which arises by addition of a multiple of the scalar curvature τ, is conformally covariant, i.e.,

$$e^{(\frac{n}{2}+1)\varphi} \circ P_2(e^{2\varphi}g) = P_2(g) \circ e^{(\frac{n}{2}-1)\varphi}$$

for all $\varphi \in C^\infty(M)$ (here the functions e^φ act as multiplication operators). The operator (1.1) is known as the conformal Laplacian or Yamabe operator. It plays a central role in conformal geometry and related geometric analysis. Here and throughout, we use the convention that $-\Delta$ is non-negative.

About twenty five year ago, a conformally covariant operator of the form $\Delta^2 + LOT$ was discovered independently by Paneitz [P], Eastwood-Singer [ES] and Riegert [R]; "LOT" indicates terms with fewer than four derivatives. On manifolds of dimension $n \geq 3$, it is defined by

$$P_4 = \Delta^2 + \delta((n-2)\mathsf{J}g - 4\mathsf{P})\#d + \left(\frac{n}{2}-2\right)\left(\frac{n}{2}\mathsf{J}^2 - 2|\mathsf{P}|^2 - \Delta\mathsf{J}\right), \tag{1.2}$$

where P is the Schouten tensor, i.e., $(n-2)\mathsf{P} = \mathrm{Ric} - \mathsf{J}g$, $\#$ denotes the natural action of symmetric bilinear forms on 1-forms and δ is the formal adjoint of the differential d. P_4 satisfies the transformation law

$$e^{(\frac{n}{2}+2)\varphi} \circ P_4(e^{2\varphi}g) = P_4(g) \circ e^{(\frac{n}{2}-2)\varphi}, \ \varphi \in C^\infty(M).$$

A significant difference between (1.1) and (1.2) is the appearance of the Ricci tensor in the Paneitz operator. The scalar curvature quantity

$$Q_4 = \frac{n}{2}\mathsf{J}^2 - 2|\mathsf{P}|^2 - \Delta\mathsf{J} \tag{1.3}$$

in the constant term of P_4 is a special case of Branson's Q-curvature [B2]. For $n=4$, the fourth-order curvature quantity Q_4 satisfies the remarkable transformation law [BO]

$$e^{4\varphi}Q_4(e^{2\varphi}g) = Q_4(g) + P_4(g)(\varphi), \tag{1.4}$$

which generalizes

$$e^{2\varphi}Q_2(e^{2\varphi}g) = Q_2(g) - P_2(g)(\varphi) \tag{1.5}$$

for $Q_2 = \mathsf{J} = \tau/2$ in dimension 2. Since $\tau/2$ is the Gauß curvature and $P_2 = \Delta$, (1.5) is nothing else than the Gauß curvature prescription equation. The Q-curvature prescription equation (1.4) has been at the center of much research in recent years (see [M] for a review).

The discovery of P_4 naturally raised the problem of constructing higher order analogs, i.e., of similarly correcting any power Δ^N of the Laplacian by appropriate lower order terms so that the resulting operator becomes conformally covariant. For $N=3$, such results are already contained in [B1]. The construction in [GJMS] of conformally covariant powers of the Laplacian in terms of the powers of the Laplacian for the Fefferman-Graham ambient metric [FG1], [FG2] settled the existence problem. In addition, it revealed obstructions to their existence on even dimensional manifolds. In the following, we shall refer to the operators constructed in [GJMS] as the GJMS-operators, and denote them by P_{2N}. On a manifold of even dimension n, the GJMS-operator of order n will be called the *critical* GJMS-operator. For more details we refer to Section 2.

The Yamabe operator (1.1) and the Paneitz operator (1.2) are the first two GJMS-operators. For higher orders, the *structure* of the GJMS-operators remained obscure up to now, and it is generally believed that explicit formulas for them are hopelessly complicated due to the exponential increase of their complexity as a function of the order. It is tempting to compare this with the complexity of heat kernels.

One of the remarkable properties of the GJMS-operators is that, through conformal variation, they are determined by their constant terms. More precisely,

$$(d/dt)|_0 \left(e^{2t\varphi} Q_{2N}(e^{2t\varphi}g)\right) = (-1)^N P_{2N}^0(g)(\varphi), \tag{1.6}$$

where

$$P_{2N}(1) = (-1)^N \left(\frac{n}{2} - N\right) Q_{2N},$$

and P_{2N}^0 denotes the non-constant part of P_{2N}. We shall refer to the quantities Q_{2N} as Branson's Q-curvatures (see Section 2) although sometimes only the critical Q-curvature Q_n bears that name. Thus, an understanding of the GJMS-operators is intimately connected with an understanding of the structure of the Q-curvatures. Since the complexity of Q-curvatures exponentially increases as well, it is generally believed that aiming for explicit formulas for high order Q-curvatures is also hopeless. Even in the presence of well-structured formulas for Q-curvatures, it remains a non-trivial problem to derive such formulas for the corresponding GJMS-operators by conformal variation.

On the other hand, motivated by the rich results in geometric analysis around P_4 and Q_4, explicit formulas for high order Q-curvatures and GJMS-operators are of substantial interest. Uncovering their structure could open the way to future geometric applications. Presently, this is an almost unexplored area.

A remarkable exception is the work [GoP]. It addresses the problem to find explicit formulas for GJMS-operators from the point of view of tractor calculus. Gover and Peterson describe an algorithm for deriving explicit formulas for these operators in terms of tractor constructions. An evaluation of the algorithm for P_8 in terms of the Levi-Civita connection and its curvature generates an explicit formula which already occupies several pages. In the opinion of the present author, these results supported the belief that the structures of the operators P_{2N} (and the related Q-curvatures Q_{2N}) are hopelessly complicated.

The moral of the present paper is that, in contrast to the accepted opinion, the complexity of high order GJMS-operators and Q-curvatures is strongly tamed by beautiful recursive structures. More precisely, we formulate systems of conjectural recursive relations among GJMS-operators and Q-curvatures, and describe how these would lead to explicit formulas. The relations are summarized in Conjecture 4.1, Conjecture 9.1 and Conjecture 9.2.

Conjecture 4.1 is supported by complete proofs of the corresponding formulas for P_6 (in general dimensions and for general metrics) and P_8 (in the critical dimension and for general metrics) as well as by basic structural results along the lines of these conjectures. We prove that the GJMS-operators on the conformally flat round spheres $\mathbb{S}^n$ are captured by the recursive algorithm and confirm an extension of that result to the conformally flat pseudo-spheres $\mathbb{S}^{q,p}$. Similarly, Conjecture 9.1 and Conjecture 9.2 are supported by proofs for Q_{2N} with $N \leq 4$ for general metrics and proofs for all spheres $\mathbb{S}^n$ and pseudo-spheres $\mathbb{S}^{q,p}$.

Although a complete understanding of the picture requires much more efforts, it is tantalizing to regard its overall simplicity as an argument in its favor.

The main features of the proposed recursive formulas for GJMS-operators are the following.

- Any GJMS-operator is described by a primary part, a secondary part (given by a second-order operator), and a constant term (given by Q-curvature).
- The primary parts are defined in terms of universal linear combinations of compositions of respective lower order GJMS-operators.

Here universality means that the coefficients of the linear combinations do not depend on the dimension of the underlying manifold.

In order to complete the description of GJMS-operators, we propose a recursive description of Q-curvatures along similar lines. The main features of the proposed recursive formulas for Q-curvatures are the following.

- Any Q-curvature is described as the sum of a primary and a secondary part.
- The primary parts of GJMS-operators and Q-curvatures are linked to each other.
- The secondary parts of Q-curvatures are given by universal formulas in terms of holographic coefficients.

The holographic coefficients (or renormalized volume coefficients [G2], [G4]) are functionals of a metric which arise as the coefficients in the Taylor expansion of the volume form of an associated Poincaré-Einstein metric. They are locally determined by the metric and can be written in terms of (derivatives of) the curvature tensor. One of these quantities plays the role of a conformal anomaly of the renormalized volume. The latter concept was introduced in connection with the AdS/CFT duality [HS], [W], [G2], [A1]. For more information see Section 9 and [J1].

In connection with the recursive formulas for Q-curvatures, the principle of universality means that formulas in the critical dimension literally hold true also in the subcritical cases.

We illustrate the recursive structure of GJMS-operators by means of the conformally covariant third power P_6 of the Laplacian. For this purpose, we restrict to locally conformally flat metrics and comment only briefly on the general case. First of all, on locally conformally flat manifolds of dimension $n = 6$, the self-adjoint operator

$$\mathbf{P}_6 \stackrel{\text{def}}{=} \left[2(P_2P_4 + P_4P_2) - 3P_2^3\right]^0 - 48\delta(\mathsf{P}^2\#d) = \Delta^3 + LOT \tag{1.7}$$

is conformally covariant, i.e., satisfies

$$e^{6\varphi}\mathbf{P}_6(e^{2\varphi}g) = \mathbf{P}_6(g)$$

for all $\varphi \in C^\infty(M)$. Here $[\cdot]^0$ denotes the non-constant part of the respective operator in brackets. Moreover, the operator $\mathbf{P}_6$ coincides with the critical GJMS-operator P_6. These results were first obtained in [J1]. An alternative proof of the conformal covariance of a generalization of $\mathbf{P}_6$ for general metrics will be given Section 13.2.

Of course, the formula (1.7) is not explicit in terms of the Levi-Civita connection and its curvature. But such formulas easily follow from (1.7) by using the formulas (1.1) and (1.2) for P_2 and P_4. Although the resulting expressions might be interesting in connection with applications to geometric analysis, they will hide the recursive structure expressed by (1.7).

Now the right-hand side of (1.7) is the sum of the non-constant part of the *primary part*

$$\mathcal{P}_6 = 2P_2P_4 + 2P_4P_2 - 3P_2^3 \tag{1.8}$$

and the *secondary part*, which is a multiple of the second-order operator $\delta(\mathsf{P}^2\#d)$; the constant term vanishes in this case. For general metrics, the secondary part contains an additional second-order operator which is defined by the Bach tensor (or rather Graham's [G4] first extended obstruction tensor).

In the locally conformally flat category, Theorem 5.1 yields a similar formula

$$\mathbf{P}_8 \stackrel{\text{def}}{=} \mathcal{P}_8^0 - c_4\delta(\mathsf{P}^3\#d),\ c_4 = 3!4!2^3$$

for a conformally covariant operator of the form $\Delta^4 + LOT$ in dimension $n = 8$. Here the primary part $\mathcal{P}_8$ is a certain linear combination of all possible compositions of lower order GJMS-operators to an operator of order 8. The secondary part is a second-order operator which is determined by the Schouten tensor P. For general metrics, Theorem 11.1 shows that the corresponding secondary parts involve additional contributions of the first two extended obstruction tensors $\Omega^{(1)}$ and $\Omega^{(2)}$.

The last point is worth emphasizing. The operators P_2, P_4 and P_6 are generated by linear combinations of compositions of respective lower order relatives and addition of suitable second-order correction terms. In particular, these constructions do not involve the Fefferman-Graham ambient metric. But in $\mathbf{P}_8$ the ambient metric is *forced* to appear in terms of the extended obstruction tensors.

It remains open whether $\mathbf{P}_8$ coincides with P_8.

Conjecture 4.1 specifies in which sense these results are special cases of a representation formula for *all* GJMS-operator. This conjecture concerns locally conformally

flat metrics. It states that the non-constant part of P_{2N} always can be written in the form

$$P_{2N}^0 = \mathcal{P}_{2N}^0 - c_N \delta(\mathsf{P}^{N-1} \# d), \tag{1.9}$$

where the primary part $\mathcal{P}_{2N}$ is a remarkable linear combination of compositions of GJMS-operators of order $\leq 2N-2$. For general metrics, only the secondary part is expected to become more complicated (generalizing Theorem 11.1).

The primary parts $\mathcal{P}_{2N}$ are defined in Section 2. Combining (1.9) with the formula (2.4) for the constant term of P_{2N}, yields a recursive formula for P_{2N} in terms of lower order GJMS-operators and Q-curvatures. In order to recognize the right-hand side of (1.9) as the non-constant part of a conformally covariant operator, it is a key step to prove that the conformal variation of $\mathcal{P}_{2N}$ is a *second-order* operator. This is done in Theorem 3.1. Theorem 3.2 provides an analogous treatment of the second-order secondary part in (1.9).

In the special case of $\mathbf{P}_6$ in dimension $n = 6$, the variational formula reads

$$(d/dt)|_0 \left(e^{6t\varphi} \mathcal{P}_6(e^{2t\varphi} g)\right) = 4[\mathcal{M}_4, [P_2, \varphi]] + 2[P_2, [\mathcal{M}_4, \varphi]], \tag{1.10}$$

where $\mathcal{M}_4 = P_4 - \mathcal{P}_4$ with $\mathcal{P}_4 = P_2^2$. Since $\mathcal{M}_4$ is a second-order operator, it follows that the right-hand side of (1.10) is a second-order operator. Combining (1.10) with the conformal variation law of $\delta(\mathsf{P}^2 \# d)$ (Theorem 3.2), proves the conformal covariance of $\mathbf{P}_6$.

It is natural to ask why $\mathbf{P}_6$ (defined by (1.7)) coincides with the GJMS-operator P_6. This coincidence does *not* follow from the above discussion. Instead, in the locally conformally flat case, it rests on the recursive formula

$$Q_6 = \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right] - 6(Q_4 + P_2(Q_2)) \cdot Q_2 + 48 \operatorname{tr}(\wedge^3 \mathsf{P}), \tag{1.11}$$

which expresses the order six curvature quantity Q_6 in terms of Q-curvatures and GJMS-operators of orders ≤ 4 (and the Schouten tensor P). Using conformal variation, i.e.,

$$P_6(g)(\varphi) = -(d/dt)|_0(e^{6t\varphi} Q_6(e^{2t\varphi} g)),$$

it follows that P_6 is given by (1.7) (for a detailed proof we refer to [J1], Section 6.12).

We illustrate the principles of the recursive description of Q-curvatures by means of (1.11). First of all, the sum

$$\mathcal{Q}_6 \stackrel{\text{def}}{=} -2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)$$

will be called the *primary part* of Q_6. The relation between the primary part $\mathcal{Q}_6$ of Q_6 and the primary part $\mathcal{P}_6$ (see (1.8)) of P_6 is obvious: in order to get the primary part of Q_6, one replaces the most right factor of each summand in the primary part of P_6 by the corresponding Q-curvature (up to a sign). The general case is defined in Definition 8.2.

(1.11) is a special case of Conjecture 9.1. It relates the secondary parts of Q-curvatures. In fact, the quantities $Q_4 + P_2(Q_2)$ and Q_2, which in (1.11) contribute

to the secondary part of Q_6, are natural relatives of $Q_6 - \mathcal{Q}_6$. They appear on the left-hand sides of the analogous formulas

$$Q_4 + P_2(Q_2) = -Q_2 \cdot Q_2 + 4\,\mathrm{tr}(\wedge^2 \mathsf{P}) \tag{1.12}$$

and

$$Q_2 = \mathrm{tr}(\mathsf{P}). \tag{1.13}$$

with the respective primary parts $\mathcal{Q}_4 = -P_2(Q_2)$ and $\mathcal{Q}_2 = 0$ of Q_4 and Q_2. In the other direction, the difference

$$Q_6 - \mathcal{Q}_6 = Q_6 - \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right]$$

contributes to the secondary part of Q_8.

There is an important equivalent formulation of Conjecture 9.1. It arises as follows. Using (1.12) and (1.13), the presentation (1.11) of Q_6 can be written in the alternative form

$$Q_6 - \mathcal{Q}_6 = -48(8v_6 - 4v_4v_2 + v_2^3), \tag{1.14}$$

where the holographic coefficients v_{2j} (see (9.1)) are given by

$$v_{2j} = (-2^{-1})^j\,\mathrm{tr}(\wedge^j \mathsf{P}).$$

(1.14) is an analog of

$$Q_4 - \mathcal{Q}_4 = 4(4v_4 - v_2^2). \tag{1.15}$$

The right-hand sides of (1.15) and (1.14) have the following interpretation. Let the functions $w_{2j} \in C^\infty(M)$ be defined by the formal power series expansion

$$\sqrt{v(r)} = 1 + w_2r^2 + w_4r^4 + w_6r^6 + \cdots,$$

where $v(r)$ is defined in (9.1). Then

$$Q_4 - \mathcal{Q}_4 = 2!2^3 w_4 \quad \text{and} \quad Q_6 - \mathcal{Q}_6 = -2!3!2^5 w_6. \tag{1.16}$$

The identities in (1.16) hold true in all dimensions $n \geq 3$ (in the locally conformally flat case). These are examples of universality.

It is convenient and natural to summarize the descriptions of the secondary parts $Q_{2N} - \mathcal{Q}_{2N}$ in form of the equality (Conjecture 9.2)

$$\mathcal{G}\left(\frac{r^2}{4}\right) = \sqrt{v(r)} \tag{1.17}$$

of generating functions. Here

$$\mathcal{G}(r) \stackrel{\mathrm{def}}{=} 1 + \sum_{N \geq 1} (-1)^N (Q_{2N} - \mathcal{Q}_{2N}) \frac{r^N}{N!(N-1)!}. \tag{1.18}$$

In particular, (1.17) contains the next identity

$$Q_8 - \mathcal{Q}_8 = 3!4!2^8 w_8$$

(see (9.15) and Example 8.3). Of course, for even n and general metrics, (1.17) requires to be interpreted as an identity of terminating Taylor series.

In connection with (1.17) some comments are in order. The generating function $\mathcal{G}$ encodes curvature quantities of a (pseudo)-Riemannian metric on a manifold M. The equality (1.17) relates $\mathcal{G}$ to the volume form of an associated Poincaré-Einstein metric on a space $X = (0, \varepsilon) \times M$ of *one more* dimension. The perspective of the AdS/CFT-duality [W] motivates to refer to this relation as a *holographic duality*. It is important to realize that the variable r plays fundamentally different roles on both sides of (1.17): while on the right-hand side it has the *geometric* meaning of a defining function of the boundary M of X, on the left-hand side it is only a *formal* variable of a generating function of data which live *on* the boundary M.

In addition, it is natural to regard $\mathcal{G}$ as the generating function of the *leading* coefficients $\mathcal{L}_{2N}$ of the residue polynomials $Q_{2N}^{res}(\lambda)$ (see (8.10)). The latter polynomials are defined as the constant terms of the respective residue families $D_{2N}^{res}(\lambda)$. In these terms,

$$\mathcal{G}(r) = -\sum_{N \geq 0} \mathcal{L}_{2N} \frac{r^N}{N!}. \tag{1.19}$$

This interpretation of $\mathcal{G}$ in full generality remains conjectural. The concept of residue families was introduced in [J1]. Their recursive structure and connections with GJMS-operators and Q-curvatures are the origin of all recursive relations discussed here. The basic properties of residue families are recalled in Section 8.

The paper is organized as follows. In Section 2, we recall the main properties of GJMS-operators and combine them to the operators $\mathcal{M}_{2N}$ of order $2N$. We display explicit formula for $\mathcal{M}_{2N}$ for $N \leq 5$ and prove that $\mathcal{P}_{2N}$ has leading part Δ^N. Section 3 contains the proofs of the conformal variational formulas for $\mathcal{P}_{2N}$ and $\mathcal{T}_{n/2-1}$. The conjectural recursive description of GJMS-operators P_{2N} (for locally conformally flat metrics) is formulated in Section 4. In Section 5, we use the results of Section 3 to derive a conformally covariant fourth-order power of the Laplacian (for locally conformally flat metrics). In Sections 6 and 7, we consider the specializations of Conjecture 4.1 to round spheres and pseudo-spheres. Section 6 gives a proof of a refinement for round spheres. In Section 8, we explain in which sense the definition of $\mathcal{M}_{2N}$ is inspired by the Q-polynomials of [J1]. This sets the background of the formulation of the conjectural recursive relations for Q-curvatures in Section 9. These relations appear in two equivalent forms: Conjecture 9.1 and Conjecture 9.2. In Section 9, we confirm the general picture for round spheres and pseudo-spheres. In Section 10, we explicate GJMS-operators for a related class of Riemannian metrics [GoL] with terminating Poincaré-Einstein metrics, and confirm Conjecture 4.1 for this class. In Section 11, we extend the construction of a conformally covariant fourth power in Section 5 to general metrics, and show that the result confirms a special case of Conjecture 11.1, which extends Conjecture 4.1 to general metrics. In Section 12, we collect comments on various open problems and perspectives. In Section 13 we present self-contained and detailed proofs of the conformal covariance of P_4 and P_6 in the respective critical dimensions and for general metrics. These proofs illustrate central arguments of the paper. The reader may start by reading them.

Throughout we use the notation and conventions of [J1]. Computer experiments using Mathematica had an important impact on this work. Such experiments were involved both in the tests of numerous identities and in the search for the hidden patterns. Typical instances for the interactions of theoretical and experimental work are Definition 2.1, Lemma 6.2 and Lemma 9.1. The related programming was done by Carsten Falk. The material in Section 6 emerged from a discussion with Christian Krattenthaler (Wien). It is a pleasure to thank him for allowing to present his proof of Theorem 6.1 in Section 6. Finally, I would like to thank Jesse Alt (Berlin) and Felipe Leitner (Stuttgart) for comments on the manuscript.

2. The operators $\mathcal{M}_{2N}$

We start by recalling the existence of conformally covariant powers of the Laplacian.

Theorem 2.1 ([GJMS]). *Let M be a manifold of dimension n. For even n and all integers $1 \le N \le \frac{n}{2}$, there exists a natural differential operator $P_{2N}(\cdot)$ of the form*

$$P_{2N} = \Delta^N + LOT$$

such that

$$e^{(\frac{n}{2}+N)\varphi} \circ P_{2N}(e^{2\varphi}g) = P_{2N}(g) \circ e^{(\frac{n}{2}-N)\varphi} \tag{2.1}$$

for all metrics g and all $\varphi \in C^\infty(M)$. For odd n, such operators exist for all $N \ge 1$.

More precisely, it is shown in [GJMS] how to derive such conformally covariant powers of the Laplacian from the powers of the Laplacian for the Fefferman-Graham ambient metric [FG2]. For even n and $2N > n$, this construction is obstructed by the obstructions to the existence of the ambient metric. However, the non-existence of conformally covariant operators of the form $\Delta^N + LOT$ for $2N > n$ is a deeper result. In fact, for $N > \frac{n}{2}$ it is impossible to correct Δ^N by lower order terms so that the resulting operator satisfies (2.1). The non-existence of conformally covariant cubes of the Laplacian on four-manifolds was discovered in [G1]. The general non-existence was established in [GoH].

On the other hand, for locally conformally flat metrics, all obstructions vanish, and the construction in [GJMS] yields an infinite sequence of conformally covariant operators P_{2N} in any dimension $n \ge 3$. Although in this case the ambient metric is completely determined by P, the complexity of explicit formulas for the corresponding GJMS-operators increases quickly with their order.

There are a few exceptional cases, in which simple explicit formulas are available. On the round sphere $\mathbb{S}^n$, GJMS-operators are intertwining operators for spherical principal series representations. Hence they can be derived from the standard Knapp-Stein intertwining operators. This yields the formula

$$P_{2N} = \prod_{j=\frac{n}{2}}^{\frac{n}{2}+N-1} (\Delta - j(n-1-j)). \tag{2.2}$$

The product formula (2.2) extends to Einstein metrics in the form

$$P_{2N} = \prod_{j=\frac{n}{2}}^{\frac{n}{2}+N-1} \left(\Delta - \frac{j(n-1-j)}{n(n-1)}\tau\right), \tag{2.3}$$

where τ is the constant scalar curvature of the Einstein metric. For details see [B2], [Go1], [G3], [FG2], and [J1]. These examples will serve as basic test cases of general statements.

In the following, we shall often distinguish (for even n) between the *critical* GJMS-operator P_n and the *subcritical* GJMS-operators P_{2N}, $2N < n$.

By relating the operators P_{2N} to scattering theory for Poincaré-Einstein metrics, Graham and Zworski [GZ] proved that all P_{2N} are formally self-adjoint.

Branson [B2] used the constant term of P_{2N} to define the scalar curvature quantity Q_{2N} through the formula

$$P_{2N}(1) = (-1)^N \left(\frac{n}{2} - N\right) Q_{2N}; \tag{2.4}$$

note that the sign $(-1)^N$ is caused by our convention that $-\Delta$ is the non-negative Laplacian. For even n, this defines Q_{2N} for $2N < n$, and we shall refer to these functions as to the *subcritical* Q-curvatures. Q_{2N} is a curvature quantity of order $2N$, i.e., its definition involves $2N$ derivatives of the metric. For even n, the *critical* Q-curvature Q_n arises from its subcritical relatives of order n (but in dimension $> n$) by the "limit" dimension $\to n$. For Q_4 as in (1.3), this just means to set $n = 4$.

Similarly, as for Q_2 and Q_4, the critical Q-curvature Q_n satisfies the fundamental linear transformation law

$$e^{n\varphi} Q_n(e^{2\varphi} g) = Q_n(g) + (-1)^{\frac{n}{2}} P_n(g)(\varphi), \tag{2.5}$$

which involves the critical GJMS-operator P_n. It shows the remarkable fact that the operator P_n is completely determined by the scalar curvature quantity Q_n. In the subcritical cases, the non-constant part P_{2N}^0 of P_{2N} is determined by the conformal variation of Q_{2N} (see (1.6)). However, the subcritical Q-curvatures do not obey a linear conformal transformation law.

Now the operators P_{2N} give rise to a sequence of operators $\mathcal{M}_{2N}$, $N \ge 1$. As for P_{2N}, this sequences is infinite in odd dimensions and possibly obstructed at $2N = n$ in even dimension n. These restrictions are in force throughout and are suppressed in the following.

We introduce some notation. A sequence $I = (I_1, \dots, I_r)$ of integers $I_j \ge 1$ will be regarded as a composition of the sum $|I| = I_1 + I_2 + \cdots + I_r$. Compositions are partitions in which the order of the summands is considered. $|I|$ will be called the size of I. We set

$$P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}.$$

Definition 2.1. *For $N \ge 1$, let*

$$\mathcal{M}_{2N} = \sum_{|I|=N} m_I P_{2I} \tag{2.6}$$

with

$$m_I = -(-1)^r |I|!(|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j!(I_j-1)!} \prod_{j=1}^{r-1} \frac{1}{I_j+I_{j+1}}. \tag{2.7}$$

Then $m_{(N)} = 1$, *and we define the primary part* $\mathcal{P}_{2N}$ *by the decomposition*

$$\mathcal{M}_{2N} = P_{2N} - \mathcal{P}_{2N}. \tag{2.8}$$

Note that

$$m_I = -(-1)^r \binom{|I|}{I_1,\dots,I_r} \binom{|I|-r}{I_1-1,\dots,I_r-1} \frac{(N-1)\cdots(N-r+1)}{\prod_{j=1}^{r-1}(I_j+I_{j+1})}$$

in terms of multinomial coefficients. Although it will not be important in the sequel, it would be interesting to know whether all m_I are integers and whether m_I has a combinatorial meaning. For compositions I with two entries, we easily find

$$m_{(I_1,I_2)} = -\binom{N-1}{I_1}\binom{N-1}{I_2} \in \mathbb{Z},\ N = I_1 + I_2. \tag{2.9}$$

The sum in (2.6) runs over all compositions I of size $|I| = N$. It contains 2^{N-1} terms. More precisely, there are exactly $\binom{N-1}{r-1}$ terms with r factors. For more details on compositions see [A2], Chapter 4.

Since $m_I \neq 0$ for all I, each possible composition of GJMS-operators to an operator of order $2N$ contributes non-trivially to the sum (2.6). Obstructions to the existence of $\mathcal{M}_{2N}$ arise only through obstructions to the existence of the GJMS-operators.

In general, GJMS-operators do not commute, and the coefficients m_I depend on the ordering of the entries of the composition I. This is the reason for the consideration of compositions instead of partitions. However, we observe

Corollary 2.1. $m_I = m_{I^{-1}}$ *for all* I, *where* $I^{-1} = (I_r, \dots, I_1)$ *denotes the reversed (or inverse) composition of* $I = (I_1, \dots, I_r)$. *In particular, all* $\mathcal{M}_{2N}$ *are self-adjoint.*

Proof. The claimed symmetry of the coefficients is obvious from (2.7). Therefore, the self-adjointness of all GJMS-operators implies the self-adjointness of all $\mathcal{M}_{2N}$. □

We display the first few operators $\mathcal{M}_{2N}$. The first two cases are very simple.

Example 2.1. $\mathcal{M}_2 = P_2$ *and* $\mathcal{M}_4 = P_4 - P_2^2$. *The corresponding primary parts are*

$$\mathcal{P}_2 = 0 \quad \text{and} \quad \mathcal{P}_4 = P_2^2. \tag{2.10}$$

The following two cases will play a substantial role in what follows.

Example 2.2.

$$\mathcal{M}_6 = P_6 - \mathcal{P}_6$$

with the primary part

$$\mathcal{P}_6 = 2(P_2P_4 + P_4P_2) - 3P_2^3. \tag{2.11}$$

Example 2.3.

$$\mathcal{M}_8 = P_8 - \mathcal{P}_8$$

with the primary part

$$\mathcal{P}_8 = (3P_2P_6 + 3P_6P_2 + 9P_4^2) - (12P_2^2P_4 + 12P_4P_2^2 + 8P_2P_4P_2) + 18P_2^4. \tag{2.12}$$

The sum contains 7 *terms.*

The following formula illustrates the exponentially increasing complexity of the situation.

Example 2.4.

$$\mathcal{M}_{10} = P_{10} - \mathcal{P}_{10}$$

with the primary part

$$\begin{aligned} \mathcal{P}_{10} = {} & (4P_2P_8 + 4P_8P_2 + 24P_4P_6 + 24P_6P_4) \\ & - (60P_2P_4^2 + 60P_4^2P_2 + 30P_2^2P_6 + 30P_6P_2^2 + 15P_2P_6P_2 + 80P_4P_2P_4) \\ & + (120P_2^3P_4 + 120P_4P_2^3 + 80P_2^2P_4P_2 + 80P_2P_4P_2^2) - 180P_2^5. \end{aligned} \tag{2.13}$$

The sum contains 15 *terms.*

The operator $\mathcal{P}_{2N}$ will play the role of a primary part of the GJMS-operator P_{2N} in the sense that it differs from P_{2N} only by a second-order operator. The following result is the minimal requirement in order to qualify $\mathcal{P}_{2N}$ for that role.

Lemma 2.1. *For all $N \geq 2$, the operator $\mathcal{P}_{2N}$ is of the form $\Delta^N + LOT$.*

Proof. The assertion is equivalent to

$$\sum_{|I|=N} m_I = 0. \tag{2.14}$$

(2.14) follows from the stronger relations

$$\sum_{J,\, a+|J|=N} m_{(a,J)} = (-1)^{N-a} \binom{N-1}{a-1} \tag{2.15}$$

for $1 \leq a \leq N$. In fact, (2.15) implies

$$\begin{aligned} (-1)^{N-1} \sum_{|I|=N} m_I &= (-1)^{N-1} \sum_{a=1}^{N} \left(\sum_{J,\, a+|J|=N} m_{(a,J)} \right) \\ &= \binom{N-1}{0} - \binom{N-1}{1} \pm \cdots + (-1)^{N-1} \binom{N-1}{N-1} = 0. \end{aligned}$$

In order to prove (2.15), we write

$$\sum_{J,\, a+|J|=N} m_{(a,J)} = \sum_{b,K,\, a+b+|K|=N} m_{(a,b,K)},$$

and note that (2.7) implies

$$m_{(a,b,K)} = -\frac{1}{a+b}\binom{N}{a}^2 \frac{a(N-a)}{N} m_{(b,K)}. \tag{2.16}$$

Hence

$$\sum_{J,\, a+|J|=N} m_{(a,J)} = -\binom{N}{a}^2 \frac{a(N-a)}{N} \sum_{b=1}^{N-a} \frac{1}{a+b} \sum_{K,\, a+b+|K|=N} m_{(b,K)}. \tag{2.17}$$

We use (2.17) to prove (2.15) by induction on N. Suppose we have already proved (2.15) up to $N-1$. Then the right-hand side of (2.17) equals

$$-\binom{N}{a}^2 \frac{a(N-a)}{N} \sum_{b=1}^{N-a} \frac{1}{a+b} (-1)^{N-a-b} \binom{N-a-1}{b-1}.$$

Thus, it suffices to verify that

$$-\binom{N-1}{a-1} = \binom{N}{a}^2 \frac{a(N-a)}{N} \sum_{b=1}^{N-a} \frac{1}{a+b} (-1)^b \binom{N-a-1}{b-1}. \tag{2.18}$$

To this end, we apply the identities

$$\sum_{j=0}^{N} \frac{1}{j+M} (-1)^j \binom{N}{j} = B(M, N+1), \ M \geq 1 \tag{2.19}$$

which follow from the formula

$$B(M, N+1) = \int_0^1 x^{M-1} (1-x)^N dx$$

for the Beta function by expanding $(1-x)^N$ as a polynomial in x and integrating term by term. (2.19) implies

$$\sum_{b=1}^{N-a} \frac{1}{a+b} (-1)^b \binom{N-a-1}{b-1} = -B(a+1, N-a) = -\frac{a!(N-a-1)!}{(N-1)!}.$$

Now a calculation shows that

$$\binom{N-1}{a-1} \Big/ \binom{N}{a}^2 \frac{a(N-a)}{N} = \frac{a!(N-a-1)!}{N!}.$$

This proves (2.18). □

The relation (2.15) will be substantially refined in Section 6.

3. Conformal variational formulas

In the present section, we prove conformal variational formulas for the operators

$$\mathcal{M}_{2N} = P_{2N} - \mathcal{P}_{2N} \quad \text{and} \quad \mathcal{T}_{n/2-1},$$

where

$$\mathcal{T}_N \stackrel{\text{def}}{=} \delta(\mathsf{P}^N \# d). \tag{3.1}$$

In (3.1), the notation does not distinguish between the symmetric bilinear form P and the induced linear operator on TM. We use this convention throughout.

The first result concerns the primary parts $\mathcal{P}_{2N}$.

Theorem 3.1. *On M^n,*

$$\begin{aligned}(d/dt)|_0\big(e^{(\frac{n}{2}+N)t\varphi}\mathcal{P}_{2N}(e^{2t\varphi}g)e^{-(\frac{n}{2}-N)t\varphi}\big) \\ = \sum_{j=1}^{N-1}\binom{N-1}{j-1}^2(N-j)\left[\mathcal{M}_{2j}(g),[\mathcal{M}_{2N-2j}(g),\varphi]\right].\end{aligned} \tag{3.2}$$

Here φ is regarded as a multiplication operator.

(3.2) holds true whenever both sides are defined. Thus, for even n and general metrics, we assume that $2N \le n$.

In the critical case, Theorem 3.1 states that

$$(d/dt)|_0\left(e^{nt\varphi}\mathcal{P}_n(e^{2t\varphi}g)\right) = \sum_{j=1}^{\frac{n}{2}-1}\binom{\frac{n}{2}-1}{j-1}^2(\frac{n}{2}-j)\left[\mathcal{M}_{2j}(g),[\mathcal{M}_{n-2j}(g),\varphi]\right].$$

Proof. The proof rests on the transformation laws (2.1). The right-hand side of (3.2) is a weighted sum of terms of the form

$$\mathcal{M}_{2j}\circ\mathcal{M}_{2N-2j}\circ\varphi - \mathcal{M}_{2j}\circ\varphi\circ\mathcal{M}_{2N-2j} - \mathcal{M}_{2N-2j}\circ\varphi\circ\mathcal{M}_{2j} + \varphi\circ\mathcal{M}_{2N-2j}\circ\mathcal{M}_{2j}.$$

On the left-hand side of (3.2), the term P_{2I} with $I = (I_1, I_2, \dots, I_r)$ in the sum $\mathcal{P}_{2N}$ induces a constant multiple of the contribution

$$\begin{aligned}(N-I_1)\varphi\circ P_{2I} - (I_1+I_2)P_{2I_1}\circ\varphi\circ P_{2I_2}\cdots P_{2I_r} \\ - \cdots - (I_{r-1}+I_r)P_{2I_1}\cdots P_{2I_{r-1}}\circ\varphi\circ P_{2I_r} + (N-I_r)P_{2I}\circ\varphi.\end{aligned} \tag{3.3}$$

For the term $P_{2I}\circ\varphi$, the claim is

$$\begin{aligned}-(N-I_r)m_I = \binom{N-1}{I_1-1}^2(N-I_1)\, m_{(I_1)}m_{(I_2,\dots,I_r)} \\ + \binom{N-1}{I_1+I_2-1}^2(N-I_1-I_2)\, m_{(I_1,I_2)}m_{(I_3,\dots,I_r)} \\ + \cdots + \binom{N-1}{I_1+I_2+\cdots+I_{r-1}-1}^2(N-I_1-I_2-\cdots-I_{r-1})\, m_{(I_1,I_2,\dots,I_{r-1})}m_{(I_r)}.\end{aligned} \tag{3.4}$$

In order to prove this identity, we use the explicit formula for the coefficients m_I (see (2.7)) to write the terms in the sum as multiples of m_I. We find

$$-\frac{1}{N}\Big[I_1(I_1+I_2)+(I_1+I_2)(I_2+I_3)+(I_1+I_2+I_3)(I_3+I_4) \\ +\cdots+(I_1+I_2+\cdots+I_{r-1})(I_{r-1}+I_r)\Big]m_I. \tag{3.5}$$

Now the relation

$$I_1(I_1+I_2)+\cdots+(I_1+I_2+\cdots+I_{r-1})(I_{r-1}+I_r)=(I_1+\cdots+I_r)(I_1+\cdots+I_{r-1})$$

(which follows by induction) implies that in (3.5) the sum in brackets equals $N(N-I_r)$ if $|I|=N$. Thus, (3.5) equals $-(N-I_r)m_I$. This proves the assertion.

Next, for the term $\varphi\circ P_{2I}$ with $|I|=N$, the claim is

$$-(N-I_1)m_I=\binom{N-1}{I_2+\cdots+I_r-1}^2(N-I_2-\cdots-I_r)\,m_{(I_1)}m_{(I_2,I_2,\dots,I_r)} \\ +\cdots+\binom{N-1}{I_r-1}^2(N-I_r)\,m_{(I_1,\dots,I_{r-1})}m_{(I_r)}.$$

This identity follows by applying (3.4) to the inverse composition I^{-1} of I and using the relations $m_{I^{-1}}=m_I$ for all compositions I (see Corollary 2.1).

It remains to prove the corresponding identities for the coefficients of the terms

$$P_{2I_1}\cdots P_{2I_a}\circ\varphi\circ P_{2I_{a+1}}\dots P_{2I_r}.$$

In that case, the claim is

$$-(I_a+I_{a+1})m_I=\Bigg[\binom{N-1}{I_1+\cdots+I_a-1}^2(N-I_1-\cdots-I_a) \\ +\binom{N-1}{I_{a+1}+\cdots+I_r-1}^2(N-I_{a+1}-\cdots-I_r)\Bigg]m_{(I_1,\dots,I_a)}m_{(I_{a+1},\dots,I_r)}.$$

By (2.7), the right-hand side reduces to

$$-\frac{1}{N}(I_a+I_{a+1})[(I_1+\cdots+I_a)+(N-I_1-\cdots-I_a)]m_I,$$

i.e., to $-(I_a+I_{a+1)})m_I$. This completes the proof. ☐

Corollary 3.1. *In the situation of Theorem 3.1,*

$$(d/dt)|_0\left(e^{(\frac{n}{2}+N)t\varphi}\mathcal{V}_{2N}(e^{2t\varphi}g)e^{-(\frac{n}{2}-N)t\varphi}\right)=\sum_{j=1}^{N-1}\frac{1}{N-j}\left[\mathcal{V}_{2j}(g),[\mathcal{V}_{2N-2j}(g),\varphi]\right],$$

where

$$\mathcal{V}_{2k}\stackrel{def}{=}-\frac{\mathcal{M}_{2k}}{(k-1)!(k-1)!}. \tag{3.6}$$

Proof. By $\mathcal{M}_{2N}=P_{2N}-\mathcal{P}_{2N}$ and the conformal covariance (2.1) of P_{2N}, the assertion is equivalent to Theorem 3.1. ☐

The self-adjointness of $\mathcal{V}_{2N}$ (Corollary 2.1) implies the self-adjointness of the conformal variation

$$(d/dt)|_0\big(e^{(\frac{n}{2}+N)t\varphi}\mathcal{V}_{2N}(e^{2t\varphi}g)e^{-(\frac{n}{2}-N)t\varphi}\big).$$

In fact, for $u,v \in C^\infty(M)$ with compact support,

$$\begin{aligned}
&\int_M (d/dt)|_0 \left(e^{(\frac{n}{2}+N)t\varphi}\mathcal{V}_{2N}(e^{2t\varphi}g)(e^{-(\frac{n}{2}-N)t\varphi}u)\right)\bar{v}\, vol(g)\\
&= (d/dt)|_0 \int_M e^{(-\frac{n}{2}+N)t\varphi}\mathcal{V}_{2N}(e^{2t\varphi}g)(e^{-(\frac{n}{2}-N)t\varphi}u)\bar{v}\, vol(e^{2t\varphi}g)\\
&= (d/dt)|_0 \int_M u\overline{e^{-(\frac{n}{2}-N)t\varphi}\mathcal{V}_{2N}(e^{2t\varphi}g)(e^{-(\frac{n}{2}-N)t\varphi}v)}\, vol(e^{2t\varphi}g)\\
&= \int_M u\overline{(d/dt)|_0 \left(e^{(\frac{n}{2}+N)t\varphi}\mathcal{V}_{2N}(e^{2t\varphi}g)(e^{-(\frac{n}{2}-N)t\varphi})v\right)}\, vol(g).
\end{aligned}$$

Corollary 3.1 confirms this observation for $\mathcal{V}_{2N}$ by using the self-adjointness of all lower order $\mathcal{V}_{2M}$, $M<N$.

The second conformal variational formula concerns the operator $\mathcal{T}_{n/2-1}$ on manifolds M^n of even dimension. Note that $\mathcal{T}_0 = -\Delta$.

Theorem 3.2. *For a locally conformally flat metric g,*

$$n(d/dt)|_0\left(e^{nt\varphi}\mathcal{T}_{\frac{n}{2}-1}(e^{2t\varphi}g)\right) = \sum_{j=1}^{\frac{n}{2}-1} j[\mathcal{T}_{j-1}(g),[\mathcal{T}_{\frac{n}{2}-1-j}(g),\varphi]]^0. \tag{3.7}$$

The proof of Theorem 3.2 will also show that for general metrics both sides of (3.7) differ by a second-order operator the main part of which is given by the sum of the terms

$$4k(\mathsf{P}^{k-1})_a^t(\mathsf{P}^{l-1})_c^r(\mathsf{P}^{\frac{n}{2}-1-k-l})_b^s\mathcal{C}_{trs}\varphi^c\,\mathrm{Hess}^{ab}(u) \tag{3.8}$$

for all integers $l,k \geq 1$ such that $l+k \leq \frac{n}{2}-1$. Here $\mathcal{C}$ denotes the Cotton tensor

$$\mathcal{C}(X,Y,Z) = \nabla_X(\mathsf{P})(Y,Z) - \nabla_Y(\mathsf{P})(X,Z). \tag{3.9}$$

We recall that $\mathcal{C}$ vanishes if the Weyl tensor C vanishes. This result will be used in Section 11.

Proof. The assertion relates two self-adjoint second-order differential operators which annihilate constants. Hence it suffices to prove that the main parts of both operators coincide. Now a calculation (using $\delta(T\#d) = -(T,\mathrm{Hess})+(\delta(T),d)$ for any symmetric bilinear form T) shows that the main part of the operator

$$[\mathcal{T}_p,[\mathcal{T}_q,\varphi]]\,,\ p,q\geq 0$$

is of the form

$$\begin{aligned}
&4(\mathsf{P}^p)_{ij}(\mathsf{P}^q)^{rs}\,\mathrm{Hess}_r^i(\varphi)\,\mathrm{Hess}_s^j(\varphi)\\
&\qquad + 4(\mathsf{P}^p)_{ij}\nabla^i(\mathsf{P}^q)^{rs}\varphi_r\,\mathrm{Hess}_s^j(u) - 2(\mathsf{P}^q)_{ij}\nabla^j(\mathsf{P}^p)_{rs}\varphi^i\,\mathrm{Hess}^{rs}(u).
\end{aligned} \tag{3.10}$$

Hence the right-hand side of (3.7) equals

$$4\sum_{k=1}^{\frac{n}{2}-1} k(\mathsf{P}^{k-1})_{ij}(\mathsf{P}^{\frac{n}{2}-1-k})^{rs}\operatorname{Hess}^j_r(\varphi)\operatorname{Hess}^j_s(u),$$

up to terms with first order derivatives of P. The latter sum simplifies to

$$n\sum_{k=1}^{\frac{n}{2}-1}(\mathsf{P}^{k-1})_{ij}(\mathsf{P}^{\frac{n}{2}-1-k})^{rs}\operatorname{Hess}^i_r(\varphi)\operatorname{Hess}^j_s(u). \tag{3.11}$$

We compare (3.11) with the main part of the left-hand side of (3.7). It is given by

$$-\sum_{k=1}^{\frac{n}{2}-1}(d/dt)|_0\left((\mathsf{P}^{k-1})_{ir}(\mathsf{P}^r_s - t\operatorname{Hess}^r_s(\varphi))(\mathsf{P}^{\frac{n}{2}-1-k})^s_j\right)\operatorname{Hess}^{ij}(u)$$
$$= \sum_{k=1}^{\frac{n}{2}-1}(\mathsf{P}^{k-1})_{ij}(\mathsf{P}^{\frac{n}{2}-1-k})^{rs}\operatorname{Hess}^j_s(\varphi)\operatorname{Hess}^i_r(u).$$

Therefore, it only remains to prove that, in the locally conformally flat case, i.e., if $\mathsf{C}=0$, the terms with derivatives of P cancel. By (3.10), these contribute

$$4\sum_{k=1}^{\frac{n}{2}-1} k(\mathsf{P}^{k-1})_{ia}\nabla^i(\mathsf{P}^{\frac{n}{2}-1-k})^{cb}\varphi^c\operatorname{Hess}^{ab}(u)$$
$$-2\sum_{k=1}^{\frac{n}{2}-1} k(\mathsf{P}^{\frac{n}{2}-1-k})_{cj}\nabla^j(\mathsf{P}^{k-1})_{ab}\varphi^c\operatorname{Hess}^{ab}(u).$$

Reordering the second sum yields

$$\left(4\sum_{k=1}^{\frac{n}{2}-2} k\mathsf{P}^{k-1}_{ia}\nabla^i(\mathsf{P}^{\frac{n}{2}-1-k})^{cb} - 2\sum_{k=1}^{\frac{n}{2}-2}\left(\frac{n}{2}-k\right)(\mathsf{P}^{k-1})_{ic}\nabla^i(\mathsf{P}^{\frac{n}{2}-1-k})^{ab}\right)\varphi^c\operatorname{Hess}^{ab}(u).$$

Now we group the terms in this sum as follows. The product rule turns the derivatives of powers of P into sums of products which contain one derivative of P. We match the resulting sum in the k^{th} term in the first sum with the sum of the respective k^{th} terms in the individual contributions in the second sum. This gives

$$\sum_{k=1}^{\frac{n}{2}-2}\Big\{\sum_{l=1}^{\frac{n}{2}-1-k} 4k(\mathsf{P}^{k-1})_{ia}(\mathsf{P}^{l-1})_{cr}\nabla^i(\mathsf{P})^r_s(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b$$
$$-2\sum_{l=1}^{\frac{n}{2}-1-k}(\frac{n}{2}-l)(\mathsf{P}^{l-1})_{ic}(\mathsf{P}^{k-1})_{ar}\nabla^i(\mathsf{P})^r_s(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b\Big\}.$$

Interchanging the roles of i and r in the second sum, yields

$$\sum_{k=1}^{\frac{n}{2}-2}\Big\{\sum_{l=1}^{\frac{n}{2}-1-k} 4k(\mathsf{P}^{k-1})_{ia}(\mathsf{P}^{l-1})_{cr}\nabla^i(\mathsf{P})^r_s(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b$$
$$- 2\sum_{l=1}^{\frac{n}{2}-1-k}(\frac{n}{2}-l)(\mathsf{P}^{k-1})_{ai}(\mathsf{P}^{l-1})_{rc}\nabla^r(\mathsf{P})^i_s(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b\Big\}.$$

We rewrite this sum as

$$\sum_{k=1}^{\frac{n}{2}-2}\Big\{\sum_{l=1}^{\frac{n}{2}-1-k} 4k(\mathsf{P}^{k-1})_{ia}(\mathsf{P}^{l-1})_{cr}\left[\nabla^i(\mathsf{P})^r_s-\nabla^r(\mathsf{P})^i_s\right](\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b$$
$$- 2\sum_{l=1}^{\frac{n}{2}-1-k}\left(\frac{n}{2}-l-2k\right)(\mathsf{P}^{k-1})_{ia}(\mathsf{P}^{l-1})_{cr}\nabla^r(\mathsf{P})^i_s(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b\Big\},$$

i.e., as

$$4\sum_{k=1}^{\frac{n}{2}-2}\sum_{l=1}^{\frac{n}{2}-1-k} k(\mathsf{P}^{k-1})^i_a(\mathsf{P}^{l-1})^r_c\mathcal{C}_{irs}(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b$$
$$- 2\sum_{k=1}^{\frac{n}{2}-2}\sum_{l=1}^{\frac{n}{2}-1-k}\left(\frac{n}{2}-l-2k\right)(\mathsf{P}^{k-1})^i_a(\mathsf{P}^{l-1})^r_c\nabla^r(\mathsf{P})_{is}(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b.$$

In the latter double sum we match, for given l, the terms for k and $k'=\frac{n}{2}-k-l$ (note that both numbers coincide iff the coefficient $\frac{n}{2}-l-2k$ vanishes). These terms are given by

$$-2\left(\frac{n}{2}-l-2k\right)(\mathsf{P}^{k-1})^i_a(\mathsf{P}^{l-1})^r_c(\mathsf{P}^{\frac{n}{2}-1-k-l})^s_b\nabla^r(\mathsf{P})_{is}$$
$$+2\left(\frac{n}{2}-l-2k\right)(\mathsf{P}^{\frac{n}{2}-1-k-l})^i_a(\mathsf{P}^{l-1})^r_c(\mathsf{P}^{k-1})^i_b\nabla^r(\mathsf{P})_{is}.$$

By summation against $\mathrm{Hess}^{ab}(u)$, these sums vanish. This yields the explicit formula (3.8) for those contributions to the main part of the right-hand side of (3.7) which contain derivatives of P. $\square$

The same argument as for $\mathcal{M}_{2N}$ shows that the conformal variation of $\mathcal{T}_{n/2-1}$ is self-adjoint. Using the self-adjointness of all $\mathcal{T}_{2M}$, $M\le\frac{n}{2}-1$, Theorem 3.2 confirms this observation for locally conformally flat metrics.

4. Universal recursive formulas for GJMS-operators

In the present section, we work in the locally conformally flat category, i.e., we assume that the Weyl tensor C vanishes. In this case, the Fefferman-Graham ambient metric terminates at the third term [FG2], and the existence of GJMS-operators is not obstructed. For comments on the general case we refer to Section 11.

The following conjecture states a recursive formula for GJMS-operators P_{2N}.

Conjecture 4.1 (Universal recursive formulas for GJMS-operators). *Let (M,g) be a locally conformally flat Riemannian manifold of dimension $n \ge 3$. Then for $N \ge 1$,*

$$\mathcal{M}_{2N}^0 = -c_N \delta(\mathsf{P}^{N-1} \# d), \tag{4.1}$$

where $c_N = 2^{N-1} N!(N-1)!$. Here $[\cdot]^0$ denotes the non-constant part of the respective operator, and δ is the negative divergence.

Some remarks are in order. First of all, since $\mathcal{M}_{2N}$ is of the form $P_{2N} - \mathcal{P}_{2N}$, the relation (4.1) is equivalent to

$$P_{2N}^0 = \mathcal{P}_{2N}^0 - c_N \delta(\mathsf{P}^{N-1} \# d) = \mathcal{P}_{2N}^0 - c_N \mathcal{T}_{N-1}.$$

This formula presents the non-constant part of P_{2N} as a linear combination of *all* products of order $2N$ which can be formed by using lower order GJMS-operators (up to the contribution $\mathcal{T}_{N-1}$).

For $\mathbb{R}^n$ with the Euclidean metric, P_{2N} coincides with Δ^N and $\mathcal{M}_{2N} = 0$ follows from the summation formula (2.14).

(4.1) combined with

$$P_{2N}(1) = (-1)^N \left(\frac{n}{2} - N\right) Q_{2N}$$

yields a recursive formula for P_{2N}. Resolving the recursion leads to a formula for P_{2N} in terms of the Q-curvatures $Q_{2N}, \dots, Q_2$ and the powers of the Schouten tensor P. The critical P_n has the special property that it only depends on the lower order Q-curvatures $Q_{n-2}, \dots, Q_2$. In turn, using recursive relations for Q-curvatures in terms of lower order GJMS-operators and lower order Q-curvatures (see Conjecture 9.1) further reduces P_{2N} step by step to lower order constructions. This method generates formulas for GJMS-operators in terms of the Schouten tensor P and its derivatives (under the assumption $\mathsf{C} = 0$).

(4.1) is called *universal* since the coefficients of $\mathcal{M}_{2N}$ do not depend on the dimension. In other words, this way of writing the non-constant part of the critical GJMS-operator literally extends to the non-critical cases.

The recursive formula (4.1) is expected to extend also to the pseudo-Riemannian case (see Section 7 for the discussion of a special case).

It is natural to summarize the relations (4.1) in terms of generating functions as

$$\sum_{N \ge 1} \mathcal{V}_{2N}^0 \left(\frac{r^2}{4}\right)^{N-1} = \delta((1 - r^2/2\,\mathsf{P})^{-2} \# d), \tag{4.2}$$

where $\mathcal{V}_{2N}$ is defined by (3.6). A natural generalization of (4.2) for general metrics g will be discussed in Section 11.

Conjecture 4.1 is supported by the special cases $N = 1$, $N = 2$ and $N = 3$. For $N = 1$, (4.1) just states the obvious relation $P_2^0 = \Delta$. The following result follows from (1.2) by a direct calculation.

Theorem 4.1. *On manifolds of dimension $n \geq 3$, the Paneitz operator P_4 is given by*

$$P_4^0 = (P_2^2)^0 - 4\delta(\mathsf{P}\#d) \tag{4.3}$$

and

$$P_4(1) = \left(\frac{n}{2}-2\right) Q_4, \quad Q_4 = \frac{n}{2}\mathsf{J}^2 - 4|\mathsf{P}|^2 - \Delta\mathsf{J}.$$

In Section 13.1, we shall derive the conformal covariance of P_4 in the critical case $n = 4$ directly from (4.3).

Theorem 4.2 ([J1], Corollary 6.12.2)**.** *On manifolds of dimension $n \geq 6$, the GJMS-operator P_6 is given by*

$$P_6^0 = \left[2(P_2P_4 + P_4P_2) - 3P_2^3\right]^0 - 48\delta(\mathsf{P}^2\#d) - \frac{16}{n-4}\delta(\mathcal{B}\#d) \tag{4.4}$$

and

$$P_6(1) = -\left(\frac{n}{2}-3\right) Q_6.$$

Here the tensor

$$\mathcal{B}_{ij} = \Delta(\mathsf{P})_{ij} - \nabla^k\nabla_j(\mathsf{P})_{ik} + \mathsf{P}^{kl}\mathsf{C}_{kijl}$$

generalizes the Bach tensor in dimension 4. The Bach tensor term in (4.4) obstructs the existence of P_6 in dimension $n = 4$.

In the locally conformally flat case, (4.4) simplifies to

$$P_6^0 = \left[2(P_2P_4 + P_4P_2) - 3P_2^3\right]^0 - 48\delta(\mathsf{P}^2\#d). \tag{4.5}$$

Obviously, this formula is a special case of Conjecture 4.1.

In [J1], we used the relation

$$-P_6^0(g)(u) = (d/dt)|_0 \left(e^{6tu}Q_6(e^{2tu}g)\right)$$

to derive (4.4) from the explicit formula

$$Q_6 = \left[-2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)\right] - 6(Q_4 + P_2(Q_2))Q_2 - 3!2!2^5 v_6 \tag{4.6}$$

for Q_6. Here

$$v_6 = -\frac{1}{8}\operatorname{tr}(\wedge^3\mathsf{P}) - \frac{1}{24(n-4)}(\mathcal{B}, \mathsf{P}). \tag{4.7}$$

The discussion in Section 8 will show that (4.6) should be regarded as a special case of the recursive formula (9.8) for Q-curvatures in terms of the leading coefficients of the Q-polynomials $Q_{2N}^{res}(\lambda)$.

In Section 13.2, we present an alternative proof of the conformal covariance of P_6 in the critical dimension $n = 6$. It illustrates the argument provided by Theorem 4.3. In the special case of P_6, it derives Theorem 4.2 in dimension $n = 6$ from Theorem 4.1 and $P_2^0 = \Delta$ by using Theorem 3.1 and Theorem 3.2.

Theorem 4.3. *For a locally conformally flat metric g, the relations*

$$\mathcal{M}^0_{2N}(g) = -c_N \mathcal{T}_{N-1}(g) \quad \text{for all} \quad N = 1, \dots, \frac{n}{2} - 1 \tag{4.8}$$

imply

$$(d/dt)|_0 \left(e^{nt\varphi} (\mathcal{P}^0_n - c_{\frac{n}{2}} \mathcal{T}_{\frac{n}{2}-1})(e^{2t\varphi} g) \right) = 0. \tag{4.9}$$

Proof. Theorem 3.1 (in the critical case) implies

$$(d/dt)|_0 \left(e^{nt\varphi} \mathcal{P}_n(e^{2t\varphi} g) \right) = \sum_{j=1}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{j-1}^2 \left(\frac{n}{2} - j \right) [\mathcal{M}_{2j}(g), [\mathcal{M}_{n-2j}(g), \varphi]] .$$

The commutators with φ do not depend on the constant terms of the respective operators. Moreover, since these commutators are of first order, their commutators with any operator contain the constant term of the latter only in their constant terms. It follows that

$$(d/dt)|_0 \left(e^{nt\varphi} \mathcal{P}^0_n(e^{2t\varphi} g) \right) = \sum_{j=1}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{j-1}^2 \left(\frac{n}{2} - j \right) \left[\mathcal{M}^0_{2j}(g), \left[\mathcal{M}^0_{n-2j}(g), \varphi\right]\right]^0 .$$

Using (4.8), the right-hand side simplifies to

$$\sum_{j=1}^{\frac{n}{2}-1} \binom{\frac{n}{2}-1}{j-1}^2 \left(\frac{n}{2} - j \right) c_j \, c_{\frac{n}{2}-j} \left[\mathcal{T}_{j-1}(g), \left[\mathcal{T}_{\frac{n}{2}-j}(g), \varphi\right]\right]^0 .$$

In order to determine the variation of the term $\mathcal{T}_{\frac{n}{2}-1}$, we apply (3.7). Thus we add

$$-\frac{1}{n} c_{\frac{n}{2}} \sum_{j=1}^{\frac{n}{2}-1} j \left[\mathcal{T}_{j-1}(g), \left[\mathcal{T}_{\frac{n}{2}-j}(g), \varphi\right]\right]^0 .$$

Now a calculation shows that

$$\binom{\frac{n}{2}-1}{j-1}^2 \left(\frac{n}{2} - j \right) c_j \, c_{\frac{n}{2}-j} = \frac{1}{n} c_{\frac{n}{2}} j.$$

The proof is complete. □

The infinitesimal conformal covariance (4.9) for all metrics in the conformal class of g implies the conformal covariance

$$e^{n\varphi} \mathbf{P}_n(e^{2\varphi} g) = \mathbf{P}_n(g)$$

of

$$\mathbf{P}_n \stackrel{\text{def}}{=} \mathcal{P}^0_n - c_{\frac{n}{2}} \mathcal{T}_{\frac{n}{2}-1}. \tag{4.10}$$

In fact, we find

$$e^{n\varphi} \mathbf{P}_n(e^{2\varphi} g) - \mathbf{P}_n(g) = \int_0^1 (d/dt) \left(e^{nt\varphi} \mathbf{P}_n(e^{2t\varphi} g) \right) dt$$

$$\begin{aligned} &= \int_0^1 e^{ns\varphi}(d/dt)|_0 \left(e^{nt\varphi}\mathbf{P}_n(e^{2t\varphi}(e^{2s\varphi}g))\right) ds \\ &= 0 \end{aligned}$$

by the infinitesimal conformal covariance of $\mathbf{P}_n$ for the metrics $e^{2s\varphi}g$.

Thus, Theorem 4.3 enables us to derive (in the locally conformally flat case) the conformal covariance of $\mathbf{P}_n$ from the presentations (4.8), i.e.,

$$P_{2N}^0 = \mathcal{P}_{2N}^0 - c_N \mathcal{T}_{N-1}$$

for *all* subcritical GJMS-operators.

In the following section, we shall apply this argument to prove the conformal covariance of $\mathbf{P}_8$ (for locally conformally flat metrics). The proof of Theorem 11.1 extends the argument to general metrics.

5. A conformally covariant fourth power of the Laplacian

As an application of Theorem 4.3 we have the following construction of a conformally covariant fourth power of the Laplacian (in the locally conformally flat category).

Theorem 5.1. *In dimension $n = 8$ and for locally conformally flat metrics, the operator*

$$\mathbf{P}_8 = \mathcal{P}_8^0 - c_4\delta(\mathsf{P}^3\#d) \tag{5.1}$$

with

$$\mathcal{P}_8 = (3P_2P_6 + 3P_6P_2 + 9P_4^2) - (8P_2P_4P_2 + 12P_2^2P_4 + 12P_4P_2^2) + 18P_2^4$$

is conformally covariant, i.e.,

$$e^{8\varphi}\mathbf{P}_8(e^{2\varphi}g) = \mathbf{P}_8(g)$$

for all $\varphi \in C^\infty(M)$.

Proof. It is obvious that $\mathbf{P}_8$ is of the form $\Delta^4 + LOT$. By Theorem 4.1 and Theorem 4.2, the operators $\mathcal{M}_2^0$, $\mathcal{M}_4^0$ and $\mathcal{M}_6^0$ satisfy the relations (4.8). Theorem 4.3 implies the infinitesimal conformal invariance of $\mathbf{P}_8$. ☐

Conjecture 4.1 for $N = 4$ extends Theorem 5.1 to non-critical dimensions. It not only claims the conformal covariance of the operator $\mathbf{P}_8$, but also asserts that it coincides with P_8. For the convenience of the reader, we restate that special case in a form which also includes a description of the recursive structure of the constant term Q_8 (for more details see Section 9).

Conjecture 5.1. *On locally conformally flat manifolds of dimension $n \geq 3$, the GJMS-operator P_8 is given by*

$$P_8 = \mathcal{P}_8^0 - 3!4!2^3\delta(\mathsf{P}^3\#d) + \left(\frac{n}{2}-4\right) Q_8, \tag{5.2}$$

where

$$\mathcal{P}_8 = (3P_2P_6 + 3P_6P_2 + 9P_4^2) - (8P_2P_4P_2 + 12P_2^2P_4 + 12P_4P_2^2) + 18P_2^4$$

$$= \Delta^4 + LOT$$

and

$$Q_8 = \mathcal{Q}_8 - 12(Q_6 - \mathcal{Q}_6)Q_2 - 18(Q_4 - \mathcal{Q}_4)^2 + 4!3!2^7 v_8, \ v_8 = 2^{-4}\,\mathrm{tr}(\wedge^4 \mathsf{P}). \tag{5.3}$$

The quantities $\mathcal{Q}_4$, $\mathcal{Q}_6$ and $\mathcal{Q}_8$ are displayed in Examples 8.1, 8.2 and 8.3.

(4.4) can be derived from (4.6) by conformal variation. Therefore, it seems natural to prove that $\mathbf{P}_8$ coincides with P_8 by conformal variation of (5.3). We will return to this problem elsewhere. For an extension of Conjecture 5.1 to general metrics see Section 11.

6. Round spheres

On the round spheres $\mathbb{S}^n$, the GJMS-operators factor into second-order operators (shifted Laplacians) according to the product formula (2.2). In particular, all GJMS-operators can be written as universal polynomials in P_2:

$$P_{2N} = \prod_{j=0}^{N-1} (P_2 + j(j+1)), \ N \geq 1. \tag{6.1}$$

Thus, (2.4) yields

Corollary 6.1. *On the round sphere $\mathbb{S}^n$,*

$$Q_{2N} = \frac{n}{2} \prod_{j=1}^{N-1} \left(\frac{n}{2} - j\right)\left(\frac{n}{2} + j\right), \ N \geq 1. \tag{6.2}$$

The following refinement of Conjecture 4.1 also describes the constant terms of the operators $\mathcal{M}_{2N}$.[1]

Theorem 6.1. *On the round spheres $\mathbb{S}^n$,*

$$\mathcal{M}_{2N} = N!(N-1)!P_2, \ N \geq 1. \tag{6.3}$$

For the proof of Theorem 6.1 we split

$$\mathcal{M}_{2N} = \sum_{|I|=N} m_I P_{2I}$$

into the sum $\sum_{a=1}^N S_{(a,N)}$ of the partial sums

$$S_{(a,N)} \stackrel{\text{def}}{=} \sum_{J,\, a+|J|=N} m_{(a,J)} P_{2a} P_{2J}. \tag{6.4}$$

Lemma 6.1. *For all non-negative integers A and B,*

$$P_{2A}P_{2B} = \sum_{j=0}^{A} (-1)^j \frac{A!B!(A+B)!}{j!(A-j)!(B-j)!(A+B-j)!} P_{2(A+B-j)}. \tag{6.5}$$

[1]The proofs of Lemma 6.1, Lemma 6.2 and Lemma 6.3 are due to C. Krattenthaler.

Proof. (6.5) is equivalent to the polynomial identity

$$p_{2A}p_{2B} = \sum_{j=0}^{A}(-1)^j \frac{A!B!(A+B)!}{j!(A-j)!(B-j)!(A+B-j)!} p_{2(A+B-j)}, \tag{6.6}$$

where

$$p_{2N}(x) \stackrel{\text{def}}{=} \prod_{j=0}^{N-1}(x+j(j+1)).$$

We write

$$p_{2N}(-y(y+1)) = \prod_{j=0}^{N-1}(-y+j)(y+1+j) = (x_1)_N(x_2)_N \tag{6.7}$$

with $x_1 = -y$ and $x_2 = y+1$. Now we substitute the (far) right-hand side of (6.7) on the right-hand side of (6.6) and write the result in hypergeometric notation:

$$\sum_{j=0}^{A}(-1)^j \frac{A!B!(A+B)!}{j!(A-j)!(B-j)!(A+B-j)!}(x_1)_{A+B-j}(x_2)_{A+B-j}$$
$$= (x_1)_{A+B}(x_2)_{A+B}\ {}_3F_2\left[\begin{matrix} -A-B, -A, -B; \\ 1-A-B-x_1, 1-A-B-x_2 \end{matrix}\right].$$

The ${}_3F_2$-series can be evaluated by means of Pfaff-Saalschütz summation formula ([AAR], Theorem 2.2.6)

$${}_3F_2\left[\begin{matrix} a, b, -n; \\ c, 1+a+b-c-n \end{matrix}\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n},$$

where n is a non-negative integer. If we apply the formula (here we use $x_1+x_2 = 1$), then after little simplification we obtain

$$(x_1)_A(x_1)_B(x_2)_A(x_2)_B = (p_{2A}p_{2B})(-y(y+1)),$$

i.e., the left-hand side of (6.6). □

The following result provides a complete description of the partial sums $S_{(a,N)}$. Its proof extends the proof of Lemma 2.1.

Lemma 6.2. *On* $\mathbb{S}^n$,

$$\sum_{J,\, a+|J|=N} m_{(a,J)}P_{2J} \tag{6.8}$$
$$= \binom{N-1}{a-1}\sum_{k=0}^{N-a-1}(-1)^{N-a-k}\binom{N}{k}\frac{(N-a)!(N-a-1)!}{(N-a-k)!(N-a-k-1)!}P_{2(N-a-k)}$$

for fixed $N \geq 2$ *and* $1 \leq a \leq N-1$.

Note that (2.15) follows from (6.8) by comparing the coefficients of $\Delta^{|J|}$.

Proof. We prove the claim by induction on N. Suppose that we have already established (6.8) up to $N-1$. By (2.16), the left-hand side of (6.8) equals

$$- \frac{N!(N-1)!}{N \cdot a!(a-1)!(N-a)!(N-a-1)!} P_{2(N-a)} \\ - \frac{N!(N-1)!}{a!(a-1)!(N-a)!(N-a-1)!} \sum_{b=1}^{N-a-1} \frac{1}{a+b} \sum_{K,\, b+|K|=N-a} m_{(b,K)} P_{2b} P_{2K}.$$

If we now use the induction hypothesis, then this sum simplifies to

$$- \frac{N!(N-1)!}{N \cdot a!(a-1)!(N-a)!(N-a-1)!} P_{2(N-a)} \\ - \frac{N!(N-1)!}{a!(a-1)!(N-a)!(N-a-1)!} \sum_{b=1}^{N-a-1} \frac{1}{a+b} \binom{N-a-1}{b-1} \\ \times \sum_{k=0}^{N-a-b-1} (-1)^{N-a-b-k} \binom{N-a}{k} \frac{(N-a-b)!(N-a-b-1)!}{(N-a-b-k)!(N-a-b-k-1)!} P_{2(N-a-b-k)} P_{2b}.$$

The next step is to apply Lemma 6.1 to $P_{2(N-a-b-k)} P_{2b}$. Thus, we arrive at the expression

$$- \frac{N!(N-1)!}{N \cdot a!(a-1)!(N-a)!(N-a-1)!} P_{2(N-a)} \\ - \frac{N!(N-1)!}{a!(a-1)!} \sum_{b=1}^{N-a-1} \frac{1}{a+b} \sum_{k=0}^{N-a-b-1} (-1)^{N-a-b-k} \frac{(N-a-b-1)!}{k!(N-a-b-k-1)!} \\ \times \sum_{j=0}^{N-a-b-k} (-1)^j \frac{b}{j!(N-a-b-k-j)!(b-j)!(N-a-k-j)!} P_{2(N-a-k-j)}.$$

At this point, we make an index transformation $s = j+k$. Then the above expression can be written in the form

$$- \frac{N!(N-1)!}{N \cdot a!(a-1)!(N-a)!(N-a-1)!} P_{2(N-a)} \\ - \frac{N!(N-1)!}{a!(a-1)!} \sum_{s=0}^{N-a-1} P_{2(N-a-s)} \sum_{b=1}^{N-a-1} (-1)^{N-a-b-s} \frac{1}{a+b} \\ \times \frac{1}{(b-1)!(N-a-b-s)!(N-a-s)!} \sum_{k=0}^{s} \binom{N-a-b-1}{k} \binom{b}{s-k}.$$

The sum over k can be evaluated by means of the identity

$$\sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n}$$

(Vandermonde's convolution). Consequently, the above expression simplifies to

$$
\begin{aligned}
&- \frac{N!(N-1)!}{N \cdot a!(a-1)!(N-a)!(N-a-1)!} P_{2(N-a)} \\
&\qquad - \frac{N!(N-1)!}{a!(a-1)!} \sum_{s=0}^{N-a-1} P_{2(N-a-s)} \sum_{b=1}^{N-a-1} (-1)^{N-a-b-s} \frac{1}{a+b} \\
&\qquad \times \frac{1}{(b-1)!(N-a-b-s)!(N-a-s)!} \binom{N-a-1}{s} \\
&\qquad = - \frac{N!(N-1)!}{a!(a-1)!} \sum_{s=0}^{N-a-1} P_{2(N-a-s)} \sum_{b=1}^{N-a} (-1)^{N-a-b-s} \frac{1}{a+b} \\
&\qquad\qquad \times \frac{1}{(b-1)!(N-a-b-s)!(N-a-s)!} \binom{N-a-1}{s}.
\end{aligned}
$$

(The reader should observe the tiny difference in the summation range for b in the last line.) If we write the sum over b in hypergeometric notation, then we obtain the expression

$$
\begin{aligned}
&- \frac{N!(N-1)!}{a!(a-1)!} \sum_{s=0}^{N-a-1} P_{2(N-a-s)} \frac{1}{a+1} \\
&\qquad \times \frac{1}{(N-a-s-1)!(N-a-s)!} \binom{N-a-1}{s} {}_2F_1 \left[\begin{matrix} a+1, a+s-N+1; \\ a+2 \end{matrix} \right].
\end{aligned}
$$

The ${}_2F_1$-series can be summed by means of the Chu–Vandermonde summation formula ([AAR], Corollary 2.2.3)

$$
{}_2F_1 \left[\begin{matrix} a, -n; \\ c \end{matrix} \right] = \frac{(c-a)_n}{(c)_n},
$$

where n is a non-negative integer. After some simplification, this leads exactly to the right-hand side of (6.8). $\square$

Lemma 6.2 shows that

$$
\mathcal{M}_{2N} = (-1)^N \binom{N}{0} V_0^N + (-1)^{N-1} \binom{N}{1} V_1^N \pm \cdots + \binom{N}{N-2} V_{N-2}^N, \tag{6.9}
$$

where

$$
V_0^N \stackrel{\text{def}}{=} P_{2N} + \sum_{a=1}^{N-1} (-1)^a \binom{N-1}{a-1} P_{2a} P_{2N-2a}
$$

and

$$
V_k^N \stackrel{\text{def}}{=} \sum_{a=1}^{N-1-k} (-1)^a \binom{N-1}{a-1} \frac{(N-a)!(N-a-1)!}{(N-a-k)!(N-a-k-1)!} P_{2a} P_{2N-2k-2a}
$$

for $k = 1, \ldots, N-2$. The following result proves Theorem 6.1.

Lemma 6.3. *For all positive integers N, we have*

$$\sum_{k=0}^{N-2}(-1)^{N-k}\binom{N}{k}V_k^N = N!(N-1)!P_2. \tag{6.10}$$

Proof. The assertion is equivalent to

$$\begin{aligned}P_{2N} + \sum_{k=0}^{N}(-1)^{N-k}\binom{N}{k}&\\ \times\sum_{a=1}^{N-1-k}(-1)^a\binom{N-1}{a-1}&\frac{(N-a)!(N-a-1)!}{(N-a-k)!(N-a-k-1)!}P_{2a}P_{2N-2a-2k} = N!(N-1)!P_2\end{aligned} \tag{6.11}$$

(we apply the convention that empty sums vanish). We use Lemma 6.1 to expand $P_{2a}P_{2N-2a-2k}$. Thus, the left-hand side in (6.11) becomes

$$\begin{aligned}P_{2N} + \sum_{k=0}^{N}(-1)^{N-k}\binom{N}{k}\sum_{a=1}^{N-1-k}(-1)^a\binom{N-1}{a-1}&\frac{(N-a)!(N-a-1)!}{(N-a-k-1)!}\\ \times\sum_{j=0}^{a}(-1)^j&\frac{a!(N-k)!}{j!(a-j)!(N-a-k-j)!(N-k-j)!}P_{2(N-k-j)}.\end{aligned}$$

We do again an index transformation: we let $s = k + j$ and hence rewrite the above expression in the form

$$\begin{aligned}P_{2N} + \sum_{s=0}^{N-1}\sum_{a=1}^{N-1}(-1)^{N+s+a}P_{2(N-s)}\binom{N-1}{a-1}&\frac{N!(N-a)!}{(N-a-s)!(N-s)!}\\ &\times\sum_{k=0}^{N}\binom{N-a-1}{k}\binom{a}{s-k}.\end{aligned}$$

The sum over k can be evaluated by means of the Chu–Vandermonde summation formula. Thus, we arrive at

$$\begin{aligned}P_{2N} &+ \sum_{s=0}^{N-1}(-1)^{N+s}P_{2(N-s)}\frac{N!(N-1)!^2}{(N-s)!s!(N-s-1)!^2}\sum_{a=1}^{N-1}(-1)^a\binom{N-s-1}{a-1}\\ &= \sum_{s=0}^{N-1}(-1)^{N+s}P_{2(N-s)}\frac{N!(N-1)!^2}{(N-s)!s!(N-s-1)!^2}\sum_{a=1}^{N}(-1)^a\binom{N-s-1}{a-1}.\end{aligned}$$

(The reader should observe the tiny difference in the summation range for a in the last line.) Finally, the binomial theorem yields that the sum over a always vanishes except if $s = N - 1$. This leads directly to the right-hand side of (6.11). □

Finally, we observe that for Einstein metrics with non-vanishing scalar curvature, the formula in Conjecture 4.1 follows from Theorem 6.1 (and its analog on the real

hyperbolic space). In fact, for such metrics the GJMS-operators are given by the product formula (2.3), and a rescaling argument gives

$$\mathcal{M}_{2N} = N!(N-1)!c^{N-1}P_2, \quad c = \frac{\tau}{n(n-1)}. \tag{6.12}$$

But this relation implies

$$\mathcal{M}^0_{2N} = -2^{N-1}N!(N-1)!\delta(\mathsf{P}^{N-1}\#d)$$

by using

$$\mathsf{P} = \frac{\tau}{2n(n-1)}g.$$

It is natural to summarize these results in terms of generating functions. One should compare the following result with the version (4.2) of Conjecture 4.1.

Corollary 6.2. *For Einstein metrics,*

$$\sum_{N\geq 1} \mathcal{V}_{2N}(r^2/4)^{N-1} = \delta\left((1-r^2/2\,\mathsf{P})^{-2}\#d\right) + \left(\frac{n}{2}-1\right)\operatorname{tr}(\mathsf{P}(1-r^2/2\,\mathsf{P})^{-2}),$$

where $\mathcal{V}_{2N}$ *is defined in* (3.6)*.*

Proof. The identity $x(1-tx)^{-2} = \sum_{N\geq 1} Nt^{N-1}x^N$ shows that

$$\sum_{N\geq 1} \mathcal{V}^0_{2N}(r^2/4)^{N-1} = \sum_{N\geq 1} N\delta(\mathsf{P}^{N-1}\#d)(r^2/2)^{N-1} = \delta((1-r^2/2\,\mathsf{P})^{-2}\#d).$$

Moreover, (6.12) gives

$$\begin{aligned}
\sum_{N\geq 1} \mathcal{V}_{2N}(1)(r^2/4)^{N-1} &= \left(\frac{n}{2}-1\right)\sum_{N\geq 1} N\left(\frac{\tau}{n(n-1)}\right)^{N-1}\mathsf{J}(r^2/4)^{N-1}\\
&= \left(\frac{n}{2}-1\right)\sum_{N\geq 1} N\left(\frac{\tau}{2n(n-1)}\right)^{N} n(r^2/2)^{N-1}\\
&= \left(\frac{n}{2}-1\right)\sum_{N\geq 1} N\operatorname{tr}(\mathsf{P}^N)(r^2/2)^{N-1}\\
&= \left(\frac{n}{2}-1\right)\operatorname{tr}(\mathsf{P}(1-r^2/2\,\mathsf{P})^{-2}).
\end{aligned}$$

The proof is complete. □

7. Pseudo-spheres

Here we discuss a special case of the literal extension of Conjecture 4.1 to pseudo-Riemannian metrics. We consider the conformally flat pseudo-spheres

$$\mathbb{S}^{(q,p)} = \mathbb{S}^q \times \mathbb{S}^p,\ p\geq 1,\ q\geq 1$$

with the metrics $g_{\mathbb{S}^q} - g_{\mathbb{S}^p}$ given by the round metrics on the factors. Through this case, the theory is connected with representation theory as follows. The Yamabe operators on the round spheres have trivial kernels. But the kernel of the Yamabe operator on $\mathbb{S}^{(q,p)}$ realizes an interesting infinite-dimensional representation of $O(q+1,p+1)$.

It was analyzed in detail in [KO1] – [KO3] and [KM]. These works illustrate the interplay between conformal geometry, representation theory and classical analysis. In particular, Kobayashi and Ørsted proved

Theorem 7.1 ([KO1], Theorem 3.6.1)**.** $\ker(P_2) \neq 0$ *iff* $p+q \in 2\mathbb{N}$. *If* $\ker(P_2) \neq 0$ *and* $(p,q) \neq (1,1)$, *then the kernel is an irreducible representation of* $O(q+1,p+1)$ *with an unitarizable underlying Harish-Chandra module.*

More generally, all GJMS-operators are intertwining operators for principal series representations of $O(q+1,p+1)$ which are induced from a maximal parabolic subgroup. This fact leads to the following reformulation of results of Molčanov.

Theorem 7.2 ([B2], Theorem 6.2)**.** *On* $\mathbb{S}^{(q,p)}$, *the GJMS-operators factorize as*

$$P_{4N} = \prod_{j=1}^{N} (B+C+(2j-1))(B-C-(2j-1))(B+C-(2j-1))(B-C+(2j-1))$$

$$= \prod_{j=1}^{N} \left[(B^2-C^2)^2-2(2j-1)^2(B^2+C^2)+(2j-1)^4\right] \tag{7.1}$$

and

$$P_{4N+2} = (-B^2+C^2) \prod_{j=1}^{N} (B+C+2j)(B-C-2j)(B+C-2j)(B-C+2j)$$

$$= (-B^2+C^2) \prod_{j=1}^{N} \left[(B^2-C^2)^2-2(2j)^2(B^2+C^2)+(2j)^4\right], \tag{7.2}$$

where

$$B^2 = -\Delta_{\mathbb{S}^q} + \left(\frac{q-1}{2}\right)^2 \quad \textit{and} \quad C^2 = -\Delta_{\mathbb{S}^p} + \left(\frac{p-1}{2}\right)^2. \tag{7.3}$$

Corollary 7.1. *On* $\mathbb{S}^{(q,p)}$,

$$Q_{2N} = \prod_{j=1}^{N-1} \left(\frac{p+q}{2}+N-2j\right) \prod_{j=0}^{N-1} \left(\frac{q-p}{2}-N+1+2j\right), \ N \geq 1. \tag{7.4}$$

For $p=0$, we have $C^2 = 1/4$ and a calculation shows that the product formulas (7.1) and (7.2) specialize to (2.2). Moreover, (7.4) is easily seen to specialize to (6.2).

The following result extends Theorem 6.1.

Theorem 7.3 ([JK])**.** *On* $\mathbb{S}^{(q,p)}$,

$$\mathcal{M}_{4N} = (2N)!(2N-1)! \left(\frac{1}{2}-B^2-C^2\right), \ N \geq 1 \tag{7.5}$$

and

$$\mathcal{M}_{4N+2} = (2N+1)!(2N)!(-B^2+C^2), \ N \geq 0. \tag{7.6}$$

The proof of Theorem 7.3 rests on an extension of Lemma 6.2.

In view of $P_2 = -B^2 + C^2$, the identities (7.5) and (7.6) are equivalent to the non-linear relations

$$2\mathcal{M}_{4N} = (2N)!(2N-1)!\left(P_4 - P_2^2\right) \quad \text{and} \quad \mathcal{M}_{4N+2} = (2N+1)!(2N)!P_2 \tag{7.7}$$

of intertwining operators for $O(q+1,p+1)$.

Now Theorem 7.3 implies

$$\mathcal{M}^0_{4N} = (2N)!(2N-1)!(\Delta_{\mathbb{S}^q} + \Delta_{\mathbb{S}^p}), \; N \geq 1,$$
$$\mathcal{M}^0_{4N+2} = (2N+1)!(2N)!(\Delta_{\mathbb{S}^q} - \Delta_{\mathbb{S}^p}), \; N \geq 0.$$

But using

$$2\mathsf{P} = \begin{pmatrix} 1_q & 0 \\ 0 & -1_p \end{pmatrix},$$

these identities can be written in the form

$$\mathcal{M}^0_{4N} = -c_{2N}\delta(\mathsf{P}^{2N-1}\#d),$$
$$\mathcal{M}^0_{4N+2} = -c_{2N+1}\delta(\mathsf{P}^{2N}\#d).$$

In other words, Theorem 7.3 confirms a special case of the literal extension of Conjecture 4.1 to pseudo-Riemannian metrics.

Finally, we observe that an easy calculation using the relations

$$(q/2-1)q/2 - (p/2-1)p/2 = (q-1)^2/4 - (p-1)^2/4,$$
$$(q/2-1)q/2 + (p/2-1)p/2 = (q-1)^2/4 + (p-1)^2/4 - 1/2$$

yields the following analog of Corollary 6.2.

Corollary 7.2. *On $\mathbb{S}^{q,p}$,*

$$\sum_{N\geq 1} \mathcal{V}_{2N}\, (r^2/4)^{N-1}$$
$$= \delta((1-r^2/2\,\mathsf{P})^{-2}\#d) + \operatorname{tr}\left(\begin{pmatrix} q/2-1 & 0 \\ 0 & p/2-1 \end{pmatrix} \mathsf{P}(1-r^2/2\,\mathsf{P})^{-2}\right).$$

8. Residue polynomials

We use the GJMS-operators P_{2N} on M to define a sequence of polynomial families $P^{res}_{2N}(\lambda)$, $N \geq 1$ of differential operators on M. Their definition is motivated by the definition of the Q-polynomials

$$Q^{res}_{2N}(\lambda) \stackrel{\text{def}}{=} (-1)^{N-1} D^{res}_{2N}(\lambda)(1) \tag{8.1}$$

as the constant terms of the residue families $D^{res}_{2N}(\lambda)$ of [J1]. We describe the relation to the operators $\mathcal{M}_{2N}$.

Definition 8.1. *Let $P^{res}_2(\lambda) = P_2$, and define the families $P^{res}_{2N}(\lambda)$ for $N \geq 2$ recursively by*

$$P_{2N}^{res}(\lambda) = \prod_{k=1}^{N-1} \left(\frac{\lambda + \frac{n}{2} - 2N + k}{k} \right) P_{2N} \\ + \sum_{j=1}^{N-1} (-1)^j \prod_{\substack{k=1 \\ k \neq j}}^{N} \left(\frac{\lambda + \frac{n}{2} - 2N + k}{k - j} \right) P_{2j} P_{2N-2j}^{res} \left(-\frac{n}{2} + 2N - j \right). \tag{8.2}$$

Formula (8.2) can be regarded as a Lagrange interpolation formula.

Theorem 8.1. *$P_{2N}^{res}(\lambda)$ is the unique polynomial of degree $N-1$ which is characterized by the conditions*

$$P_{2N}^{res}\left(-\frac{n}{2} + N\right) = (-1)^{N-1} P_{2N}$$

and

$$P_{2N}^{res}\left(-\frac{n}{2} + 2N - j\right) = (-1)^j P_{2j} P_{2N-2j}^{res}\left(-\frac{n}{2} + 2N - j\right)$$

for $j = 1, \ldots, N-1$.

By resolving the recursions, we find

$$P_{2N}^{res}(\lambda) = \sum_{|I|=N} a_I(\lambda) P_{2I} \tag{8.3}$$

with polynomial coefficients $a_I(\lambda)$ of degree $N-1$.

Conjecture 8.1.

$$\mathcal{M}_{2N} = (d/d\lambda)^{N-1}|_0 (P_{2N}^{res}(\lambda)), \ N \geq 2. \tag{8.4}$$

The relation (8.4) can be confirmed by computer calculations for not too large N. Thus, under Conjecture 8.1,

$$P_{2N}^{res}(\lambda) = \mathbf{C}_{2N}^{N-1} \frac{(\lambda + \frac{n}{2} - N)^{N-1}}{(N-1)!} + \mathbf{C}_{2N}^{N-2} \frac{(\lambda + \frac{n}{2} - N)^{N-2}}{(N-2)!} + \cdots + \mathbf{C}_{2N}^{0}$$

with

$$\mathbf{C}_{2N}^{N-1} = \mathcal{M}_{2N}.$$

The coefficients are given by universal, i.e., dimension independent, linear combinations of compositions of GJMS-operators. In particular, $\mathbf{C}_{2N}^0 = (-1)^{N-1} P_{2N}$ and the critical polynomial $P_n^{res}(\lambda)$ has the form

$$P_n^{res}(\lambda) = \mathcal{M}_n \frac{\lambda^{\frac{n}{2}-1}}{(\frac{n}{2}-1)!} + \cdots + (-1)^{\frac{n}{2}-1} P_n. \tag{8.5}$$

However, only the leading coefficient and the constant term of $P_{2N}^{res}(\lambda)$ are self-adjoint.

Thus, under Conjecture 8.1, Conjecture 4.1 describes the leading coefficient $\mathbf{C}_{2N}^{N-1}$ of the polynomial $P_{2N}^{res}(\lambda)$ as a certain differential operator of *second* order. More generally, we expect that the coefficients $\mathbf{C}_{2N}^j$ are differential operators of respective orders $2N-2j$. Identifying these operators yields additional recursive formulas.

The operator $\mathcal{P}_{2N}$ describes the majority of contributions to P_{2N}. It is accompanied by a scalar curvature quantity which plays a similar role in recursive formulas for the Q-curvature Q_{2N} (see Section 9). We recall that

$$\mathcal{P}_{2N} = -\sum_{|I|=N,\, I\neq(N)} m_I P_{2I}.$$

Definition 8.2. *For $N \geq 2$, we set*

$$(-1)^N \mathcal{Q}_{2N} = -\sum_{a+|J|=N,\, a\neq N} m_{(J,a)}(-1)^a P_{2J}(Q_{2a}). \tag{8.6}$$

The first few of these curvature quantities read as follows.

Example 8.1. $\mathcal{Q}_4 = -P_2(Q_2)$.

Example 8.2. $\mathcal{Q}_6 = -2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)$.

Example 8.3.

$\mathcal{Q}_8 = -3P_2(Q_6) - 3P_6(Q_2) + 9P_4(Q_4) + 8P_2P_4(Q_2) - 12P_2^2(Q_4) + 12P_4P_2(Q_2) - 18P_2^3(Q_2)$.

The conjectural appearance of $\mathcal{P}_{2N}$ in the leading coefficient of $P_{2N}^{res}(\lambda)$ has an analog for $\mathcal{Q}_{2N}$: it is conjectured to appear in the leading coefficient $\mathcal{L}_{2N}$ of the polynomial $Q_{2N}^{res}(\lambda)$.

In the subcritical case $2N < n$, the polynomial $Q_{2N}^{res}(\lambda)$ is a polynomial of degree N which is recursively determined by the N relations

$$Q_{2N}^{res}\left(-\frac{n}{2}+2N-j\right) = (-1)^j P_{2j} Q_{2N-2j}^{res}\left(-\frac{n}{2}+2N-j\right) \tag{8.7}$$

for $j = 1, \ldots, N-1$ and

$$Q_{2N}^{res}\left(-\frac{n}{2}+N\right) = -\left(\frac{n}{2}-N\right) Q_{2N}, \tag{8.8}$$

together with either

$$Q_{2N}^{res}(0) = 0 \tag{8.9}$$

or a formula which relates $\dot{Q}_{2N}^{res}(-\frac{n}{2}+N)$ and Q_{2N}. The critical Q-polynomial $Q_n^{res}(\lambda)$ is recursively determined by the relations

$$Q_n^{res}\left(\frac{n}{2}-j\right) = (-1)^j P_{2j} Q_{n-2j}^{res}\left(\frac{n}{2}-j\right)$$

for $j = 1, \ldots, \frac{n}{2}-1$,

$$Q_n^{res}(0) = 0,$$

and the identity

$$\dot{Q}_n^{res}(0) = Q_n.$$

In full generality, these characterizations are conjectural. For background and full details on residue families we refer to [J1].

In these terms, we conjecture that

$$Q_{2N}^{res}(\lambda) = \mathcal{L}_{2N}\left(\lambda+\frac{n}{2}-N\right)^N + \cdots \tag{8.10}$$

$$= (-1)^N \left[\mathcal{Q}_{2N} - Q_{2N}\right] \frac{(\lambda+\frac{n}{2}-N)^N}{(N-1)!} + \cdots .$$

Similarly, the critical polynomial $Q_n^{res}(\lambda)$ is conjectured to have the form

$$Q_n^{res}(\lambda) = (-1)^{\frac{n}{2}} \left[\mathcal{Q}_n - Q_n\right] \frac{\lambda^{\frac{n}{2}}}{(\frac{n}{2}-1)!} + \cdots + Q_n \lambda. \tag{8.11}$$

The fact that the critical Q-curvature Q_n appears as the linear term is a consequence of the holographic formula [GJ].

We describe the role of Conjecture 8.1 and of the related conjectural relation (8.10) between $Q_{2N} - \mathcal{Q}_{2N}$ and $\mathcal{L}_{2N}$. This relation is the source of a conjectural recursive description of Q-curvatures in terms of lower order Q-curvatures and the volume of Poincaré-Einstein metrics. This will be discussed in Section 9. By conformal variation, the resulting formulas for Q-curvatures imply formulas for the corresponding GJMS-operators which naturally contain the primary parts $\mathcal{P}_{2N}$.

We illustrate the idea by considering the special case of Q_6 in the critical dimension $n = 6$. $Q_6^{res}(\lambda)$ is a polynomial of degree 3. Its characterizing properties imply that it has the form

$$Q_6^{res}(\lambda) = \frac{\lambda^3}{2!}(Q_6 + 2P_2(Q_4) - 2P_4(Q_2) + 3P_2^2(Q_2)) + \cdots ,$$

i.e.,

$$Q_6^{res}(\lambda) = \frac{\lambda^3}{2!}(Q_6 - \mathcal{Q}_6) + \cdots$$

(see Example 8.2). On the other hand, an evaluation of the definition of $Q_6^{res}(\lambda)$ as the constant term of $D_6^{res}(\lambda)$ (see the discussion in [J1], Section 6.11) shows that the quantity

$$\Lambda_6 = Q_6 - \mathcal{Q}_6$$

coincides with

$$-6(Q_4 + P_2(Q_2)) \cdot Q_2 - 2!3!2^5 v_6,$$

where v_6 is defined by (9.1). The resulting identity is a recursive formula for Q_6 in terms of P_4, P_2, Q_4, Q_2 *and* v_6. Now recall that conformal variation of Q_6 yields the non-constant part P_6^0 of P_6. By the above representation of Q_6, the resulting formula for P_6 reads

$$P_6 = (2P_2P_4 + 2P_4P_2 - 3P_2^3) + \cdots = \mathcal{P}_6 + \cdots .$$

In fact, Theorem 4.2 shows that $\mathcal{P}_6$ covers *all* but a certain second-order term.

The latter result suggests to generate similar recursive formulas for P_{2N} by conformal variation of the leading coefficient of the polynomial $Q_{2N}^{res}(\lambda)$. Through Definition 8.2, Conjecture 8.1 connects the recursively defined leading coefficient of $Q_{2N}^{res}(\lambda)$ with the definition of $\mathcal{M}_{2N}$ in Definition 2.1. Theorem 3.1 shows that the conformal variation of $\mathcal{P}_{2N}$ is only a second-order operator.

9. Universal recursive formulas for Q-curvatures

In the present section, we discuss universal recursive formulas for Q-curvatures. They describe Q_{2N} as the sum of its *primary part* $\mathcal{Q}_{2N}$ and its *secondary part*. The primary part is determined by lower order Q-curvatures and lower order GJMS-operators. We discuss two equivalent descriptions of the secondary parts, and confirm the general picture for round spheres and pseudo-spheres. A different type of recursive formulas for Q-curvatures was discussed in [FJ] and [J1].

It is natural to compare the formula in Conjecture 9.1 with the holographic formula of [GJ] which relates the critical Q-curvature Q_n of a Riemannian manifold (M,g) of even dimension n to the holographic coefficients $v_2,\dots,v_n$. We start by recalling this identity. (M,g) gives rise to a Poincaré-Einstein metric

$$g_+ = r^{-2}(dr^2+g_r),\ g_0 = g$$

on the space $(0,\varepsilon)\times M$ (for sufficiently small ε). The coefficients in the formal Taylor series

$$v(r) = \frac{vol(g_r)}{vol(g)} = 1 + v_2 r^2 + v_4 r^4 + \cdots + v_n r^n + \cdots \tag{9.1}$$

are functionals of the metric g. These are the renormalized volume coefficients of [G2], [G4] (called holographic coefficients in [J1]). For locally conformally flat metrics, the functionals v_{2j} are given by the formula

$$v_{2j} = (-1)^j \frac{1}{2^j}\operatorname{tr}(\wedge^j \mathsf{P}). \tag{9.2}$$

The functionals [V]

$$\sigma_j = \operatorname{tr}(\wedge^j \mathsf{P})$$

give rise to the so-called σ_j-Yamabe problem which, in recent years, has been studied intensively. However, in dimensions $n\ge 6$, these studies are restricted to the locally conformally flat case [BG2]. It was suggested in [CF] that for general metrics the functionals v_{2j} should be regarded as natural substitutes.

Now let u be an eigenfunction of the Laplacian of g_+, i.e., $-\Delta_{g_+}u = \lambda(n-\lambda)u$. Its formal asymptotics

$$u \sim \sum_{j\ge 0} r^{\lambda+2j}\mathcal{T}_{2j}(\lambda)(f),\ r\to 0$$

defines a sequence of rational families of differential operators $\mathcal{T}_{2j}(\lambda)$ on $C^\infty(M)$. These should not be confused with the operators $\mathcal{T}_j$ in (3.1). Let $\mathcal{T}^*_{2j}(\lambda)$ denote the formal-adjoint operator with respect to the metric g. Then [GJ]

$$(-1)^{\frac{n}{2}}\left(\left(\frac{n}{2}\right)!\left(\frac{n}{2}-1\right)!\,2^{n-1}\right)^{-1} Q_n = \frac{1}{n}\left(\sum_{j=1}^{\frac{n}{2}-1}(n-2j)\mathcal{T}^*_{2j}(0)(v_{n-2j})\right) + v_n. \tag{9.3}$$

This formulation uses the conventions of [J1]. For a discussion of an extension of (9.3) to subcritical Q-curvatures we refer to [J1], Section 6.9.

As described in Section 8, the recursive formulas for GJMS-operators in Conjecture 4.1 are suggested by recursive formulas for Q-curvatures which arise from its relation to the Q-polynomials. For Q_4 and Q_6, these recursive formulas read

$$(Q_4 - \mathcal{Q}_4) + Q_2^2 = 2!2^3 v_4 \tag{9.4}$$

and

$$(Q_6 - \mathcal{Q}_6) + 6(Q_4 - \mathcal{Q}_4)Q_2 = -2!3!2^5 v_6 \tag{9.5}$$

for

$$\mathcal{Q}_4 = -P_2(Q_2) \quad \text{and} \quad \mathcal{Q}_6 = -2P_2(Q_4) + 2P_4(Q_2) - 3P_2^2(Q_2)$$

(see Example 8.1 and Example 8.2). Next, in [J1], Section 6.13 we derived the formula

$$(Q_8 - \mathcal{Q}_8) + 12(Q_6 - \mathcal{Q}_6)Q_2 + 18(Q_4 - \mathcal{Q}_4)^2 = 3!4!2^7 v_8 \tag{9.6}$$

with $\mathcal{Q}_8$ as in Example 8.3 (for $n = 8$ and under some technical assumption which probably can be removed).

In order to formulate an extension of (9.4) – (9.6), it will be convenient to use the notation

$$\Lambda_{2j} \stackrel{\text{def}}{=} Q_{2j} - \mathcal{Q}_{2j}. \tag{9.7}$$

Conjecture 9.1 (**Universal recursive formulas for Q-curvatures**). *For even n and $2N \le n$,*

$$2\Lambda_{2N} + \sum_{j=1}^{N-1} \frac{j(N-j)}{N} \binom{N}{j}^2 \Lambda_{2N-2j}\Lambda_{2j} = (-1)^N N!(N-1)!2^{2N} v_{2N}. \tag{9.8}$$

For odd n, the same relations hold true for all N.

It should be emphasized that the recursive relations in Conjecture 9.1 are much simpler than those discussed in [FJ].

Example 9.1. *For $N = 3$ and $N = 4$,* (9.8) *reads*

$$2\Lambda_6 + 6\Lambda_2\Lambda_4 + 6\Lambda_4\Lambda_2 = -2!3!2^6 v_6$$

and

$$2\Lambda_8 + 12\Lambda_2\Lambda_6 + 36\Lambda_4^2 + 12\Lambda_6\Lambda_2 = 3!4!2^8 v_8.$$

These identities are equivalent to (9.5) *and* (9.6)*, respectively.*

It is a common feature of (9.3) and (9.8) that they describe the differences

$$Q_{2N} - (-1)^N N!(N-1)!2^{2N-1} v_{2N}.$$

But both descriptions do this in fundamentally different ways. We compare (9.8) for the critical Q-curvature with the holographic formula (9.3). Both formulas provide expressions for the difference

$$Q_n - (-1)^{\frac{n}{2}} \left(\frac{n}{2}\right)! \left(\frac{n}{2} - 1\right)! 2^{n-1} v_n.$$

The following observation points to the subtleties of the relation. For the critical Q_6 we compare (9.5), i.e.,

$$Q_6 = -2P_2(Q_4) + (2P_4 - 3P_2^2)(Q_2) - 6(Q_4 - \mathcal{Q}_4)Q_2 - 2!3!2^5 v_6 \tag{9.9}$$

with the holographic formula

$$Q_6 = -2^6 \left(4\mathcal{T}_2^*(0)(v_4) + 2\mathcal{T}_4^*(0)(v_2)\right) - 2!3!2^5 v_6. \tag{9.10}$$

The operators $\mathcal{T}_2(0)$ and $\mathcal{T}_4(0)$ have the form

$$\mathcal{T}_2(0) = 2^{-3}(\Delta + \cdots), \quad \mathcal{T}_4(0) = 2^{-6}(\Delta^2 + \cdots).$$

The term $\Delta^2 \mathsf{J} = \Delta^2 Q_2$ contributes to Q_6 with the coefficient 1. In (9.10), this terms is captured by $\mathcal{T}_4^*(0)(v_2)$ using $v_2 = -\frac{1}{2}\mathsf{J}$. On the other hand, both terms $P_2(Q_4)$ and $(2P_4 - 3P_2^2)(Q_2)$ in (9.9) are required to cover this contribution.

Note that (9.8) is compatible with the result [B2] that, in Q_{2N}, the contribution with the largest number of derivatives is $(-1)^{N-1}\Delta^{N-1}(\mathsf{J})$. In fact, the primary part $\mathcal{Q}_{2N}$ of Q_{2N} (see Definition 8.2) contains the contribution

$$\begin{aligned}(-1)^N \sum_{a+|J|=N,\, a\neq N} m_{(J,a)} \Delta^{|J|}\Delta^{a-1}(\mathsf{J}) &= (-1)^N \left(\sum_{|I|=N,\, I\neq(N)} m_I\right) \Delta^{N-1}(\mathsf{J}) \\ &= (-1)^{N-1}\Delta^{N-1}(\mathsf{J}) \qquad \text{(by (2.14))}.\end{aligned}$$

A repeated application of (9.8) yields a formula for Q_{2N} as a sum of the primary part $\mathcal{Q}_{2N}$ and a linear combination of terms of the form

$$v_{2I} = v_{2I_1} \cdots v_{2I_r}$$

for $I = (I_1, \dots, I_r)$ with $|I| = N$. Before we describe the structure of these formulas, we display the first few special cases.

Example 9.2. *The identity $Q_2 = -2v_2$ is trivial. Moreover, we have*

$$Q_4 = \mathcal{Q}_4 + 4(4v_4 - v_2^2)$$

and

$$Q_6 = \mathcal{Q}_6 - 48(8v_6 - 4v_4v_2 + v_2^3).$$

Now let

$$\mathcal{G}(r) \stackrel{\text{def}}{=} 1 + \sum_{N\geq 1} (-1)^N \Lambda_{2N} \frac{r^N}{N!(N-1)!}. \tag{9.11}$$

Ignoring the problems caused by obstructions, the following conjecture reformulates Conjecture 9.1 in terms of generating functions.

Conjecture 9.2 (Duality).

$$\mathcal{G}\left(\frac{r^2}{4}\right) = \sqrt{v(r)}, \tag{9.12}$$

where v is defined by (9.1).

In fact, (9.8) is equivalent to

$$2\frac{\Lambda_{2N}}{N!(N-1)!} + \sum_{j=1}^{N-1} \frac{\Lambda_{2N-2j}}{(N-j)!(N-1-j)!} \frac{\Lambda_{2j}}{j!(j-1)!} = (-1)^N 2^{2N} v_{2N}.$$

This yields the equivalence.

The formulation of Conjecture 9.2 can be taken literally, for instance, for locally conformally flat metrics. For general metrics and even n, the possibly existing obstructions require to interpret (9.12) as an identity of terminating Taylor series.

By definition, the generating function $\mathcal{G}$ lives on M and its variable r is a formal variable. (9.12) relates it to the volume form of an associated Poincaré-Einstein metric on a space $X = (0, \varepsilon) \times M$ of one more dimension. In this connection, the variable r has a geometric meaning. The relation (9.12) resembles the statements around the AdS/CFT-duality which, for instance, claim relations between super-string theory on AdS_5 and super-Yang-Mills theory on its boundary [W]. The common flavor motivates to refer to (9.12) as a duality. Its validity in the special case of $X = AdS_5$ with the boundary $M = \mathbb{S}^{3,1}$ is one of the facts which support the general formulation (see Corollary 9.1 and the remarks following it).

A formal calculation of the square root in (9.12) yields a power series in even powers of r with coefficients that are linear combinations of terms of the form v_{2I}. More precisely,

$$\sqrt{v(r)} = 1 + w_2 r^2 + w_4 r^4 + w_6 r^6 + \cdots , \tag{9.13}$$

where

$$\begin{aligned} 2w_2 &= v_2, \\ 2w_4 &= \frac{1}{4}\left(4v_4 - v_2^2\right), \\ 2w_6 &= \frac{1}{8}\left(8v_6 - 4v_4 v_2 + v_2^3\right). \end{aligned}$$

Now comparing coefficients in (9.12), yields

$$(-1)^N (Q_{2N} - \mathcal{Q}_{2N}) = 2^{2N} N!(N-1)! w_{2N}. \tag{9.14}$$

The first three of these identities are given in Example 9.2. The next relation

$$Q_8 - \mathcal{Q}_8 = 2^8 4!3! w_8 \tag{9.15}$$

with

$$2w_8 = \frac{1}{64}\left(64v_8 - 32v_6 v_2 - 16v_4^2 + 24v_2^2 v_4 - 5v_2^4\right)$$

is already implicit in the proof of Theorem 6.13.1 in [J1] (without the identification of w_8, however).

The relations (9.14) replace the description (9.8) of the secondary parts $Q_{2N} - \mathcal{Q}_{2N}$ by a description in terms of holographic coefficients.

Next, we prove Conjecture 9.1 and Conjecture 9.2 for the round spheres. The proofs rest on the following result.

Lemma 9.1. *On $\mathbb{S}^n$, we have*

$$\frac{\Lambda_{2N}}{(N-1)!} = \prod_{j=0}^{N-1} \left(\frac{n}{2} - j\right), \quad N \geq 1. \tag{9.16}$$

Hence

$$\mathcal{G}(r) = \sum_{N \geq 0} (-1)^N \binom{\frac{n}{2}}{N} r^N = (1-r)^{\frac{n}{2}}.$$

Now for $\mathbb{S}^n$, we have [G2]

$$g_r = (1 - r^2/4)^2 g \quad \text{and} \quad v(r) = (1 - r^2/4)^n.$$

This proves (9.12). In other words, we have proved

Theorem 9.1. *Conjecture 9.1 and Conjecture 9.2 hold true on round spheres.*

We continue with the proof of Lemma 9.1.

Proof. By Definition 8.2 and (9.7), the assertion is equivalent to

$$\sum_{a+|J|=N} m_{(J,a)} (-1)^a P_{2J}(Q_{2a}) = (-1)^N (N-1)! \prod_{j=0}^{N-1} \left(\frac{n}{2} - j\right). \tag{9.17}$$

Corollary 2.1 shows that, on general manifolds, the left-hand side of (9.17) equals

$$\begin{aligned} \sum_{a=1}^{N} (-1)^a \left(\sum_{|J|=N-a} m_{(J,a)} P_{2J} \right) (Q_{2a}) &= \sum_{a=1}^{N} (-1)^a \left(\sum_{|J|=N-a} m_{(a,J)} P_{2J^{-1}} \right) (Q_{2a}) \\ &= \sum_{a=1}^{N} (-1)^a \left(\sum_{|J|=N-a} m_{(a,J)} P_{2J} \right)^* (Q_{2a}). \end{aligned}$$

Now by Lemma 6.2, the latter sum simplifies to

$$\begin{aligned} &(-1)^N Q_{2N} + \sum_{a=1}^{N-1} (-1)^a \binom{N-1}{a-1} \\ &\times \sum_{k=0}^{N-a-1} (-1)^{N-a-k} \binom{N}{k} \frac{(N-a)!(N-a-1)!}{(N-a-k)!(N-a-k-1)!} P_{2N-2a-2k}(Q_{2a}). \end{aligned} \tag{9.18}$$

But since Q-curvatures of round spheres are constant,

$$P_{2N-2a-2k}(Q_{2a}) = (-1)^{N-a-k} \left(\frac{n}{2} - (N-a-k)\right) Q_{2N-2a-2k} Q_{2a}.$$

Hence (9.18) can be written as

$$(-1)^N Q_{2N} + \sum_{a=1}^{N-1} (-1)^a \binom{N-1}{a-1} \tag{9.19}$$

$$\times \sum_{k=0}^{N-a-1} \binom{N}{k} \frac{(N-a)!(N-a-1)!}{(N-a-k)!(N-a-k-1)!} \left(\frac{n}{2}-(N-a-k)\right) Q_{2N-2a-2k} Q_{2a}.$$

Now we express the products $Q_{2N-2a-2k}Q_{2a}$ as linear combinations of $Q's$. Taking constant terms in Lemma 6.1, yields

$$\left(\frac{n}{2}-a\right)\left(\frac{n}{2}-(N-a-k)\right) Q_{2a}Q_{2N-2a-2k} = \sum_{j=0}^{a} \frac{a!(N-a-k)!(N-k)!}{j!(a-j)!(N-a-k-j)!(N-k-j)!} \left(\frac{n}{2}-(N-k-j)\right) Q_{2N-2k-2j}.$$

By similar arguments as in the proof of Lemma 6.3, (9.19) leads to

$$(-1)^N Q_{2N} + \sum_{s=0}^{N-1}\sum_{a=1}^{N-1} (-1)^a \frac{(\frac{n}{2}-(N-s))}{(\frac{n}{2}-a)} Q_{2N-2s} \binom{N-1}{a-1} \frac{N!(N-a)!}{(N-a-s)!(N-s)!} \times \sum_{k=0}^{N} \binom{N-a-1}{k}\binom{a}{s-k}.$$

As in the proof of Lemma 6.3, the sum over k equals $\binom{N-1}{s}$, and we get

(9.20)
$$\sum_{s=0}^{N-1} \left(\frac{n}{2}-(N-s)\right) Q_{2N-2s} \frac{N!(N-1)!^2}{(N-s)!s!(N-s-1)!^2} \sum_{a=1}^{N} (-1)^a \frac{1}{(\frac{n}{2}-a)} \binom{N-s-1}{a-1}.$$

Now the well-known partial fraction expansion

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{x+k} = \frac{n!}{x(x+1)\cdots(x+n)}, \ x \neq 0, -1-, \dots, -n$$

(see [GKP], p. 188) simplifies (9.20) to

(9.21)
$$N!(N-1)!^2 \times \sum_{s=0}^{N-1} \left(\frac{n}{2}-(N-s)\right) Q_{2N-2s} \frac{1}{(-\frac{n}{2}+1)\cdots(-\frac{n}{2}+N-s)} \frac{1}{(N-s)!\, s!\, (N-s-1)!},$$

i.e.,

$$N!(N-1)!^2(-1)^N \sum_{s=0}^{N-1} (-1)^s \frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+(N-s-1))}{(N-s)!\, s!\, (N-s-1)!}. \tag{9.22}$$

In the last step, we used the product formula (see Corollary 6.1)

$$Q_{2N} = \frac{n}{2} \prod_{j=1}^{N-1} \left(\frac{n}{2}-j\right)\left(\frac{n}{2}+j\right). \tag{9.23}$$

Now we write (9.22) in the form

$$-N!(N-1)!^2\sum_{s=0}^{N-1}(-1)^s\frac{\frac{n}{2}(\frac{n}{2}+1)\cdots(\frac{n}{2}+s)}{(s+1)!\,(N-1-s)!\,s!} = -N!(N-1)!\sum_{s=0}^{N-1}(-1)^s\binom{N-1}{s}\binom{\frac{n}{2}+s}{\frac{n}{2}-1}$$

and apply the identity ([GKP], (5.24))

$$\sum_k(-1)^k\binom{l}{m+k}\binom{s+k}{n} = (-1)^{l+m}\binom{s-m}{n-l}, \quad l,m,n\in\mathbb{Z},\ l\geq 0$$

(which can be proved by induction on l). Thus, we find

$$N!(N-1)!(-1)^N\binom{\frac{n}{2}}{N} = (-1)^N(N-1)!\prod_{j=0}^{N-1}\left(\frac{n}{2}-j\right).$$

This proves (9.17). In the above arguments, we have suppressed the following subtlety in the critical case $2N=n$. The sum (9.20) involves an undefined term for $s=0$ and $a=\frac{n}{2}$. For these parameters, the fraction $(\frac{n}{2}-(N-s))/(\frac{n}{2}-a)$ is to be interpreted as 1. Likewise, in (9.21) the undefined fraction for $s=0$ is to be interpreted appropriately. The proof is complete. □

Lemma 9.1 supports the conjectural relation (8.10)

$$\mathcal{L}_{2N} = -(-1)^N\frac{\Lambda_{2N}}{(N-1)!}$$

between the secondary part $Q_{2N}-\mathcal{Q}_{2N}$ and the leading coefficient $\mathcal{L}_{2N}$ of $Q_{2N}^{res}(\lambda)$. In fact, for the sphere $\mathbb{S}^n$, the following result describes the full polynomials $Q_{2N}^{res}(\lambda)$.

Lemma 9.2. *On* $\mathbb{S}^n$*,*

$$Q_{2N}^{res}(\lambda) = -(-1)^N\frac{n}{2}\prod_{j=1}^{N-1}\left(\frac{n}{2}-j\right)\lambda\prod_{j=1}^{N-1}(\lambda-N-j)$$

for all $N\geq 1$*.*

Proof. In the non-critical case $2N\neq n$, it suffices to verify that the given expression satisfies the characterizing properties

$$Q_{2N}^{res}\left(-\frac{n}{2}+2N-j\right) = (-1)^jP_{2j}Q_{2N-2j}^{res}\left(-\frac{n}{2}+2N-j\right),\ j=1,\dots,N-1$$

and

$$Q_{2N}^{res}\left(-\frac{n}{2}+N\right) = -\left(\frac{n}{2}-N\right)Q_{2N}, \quad Q_{2N}^{res}(0)=0 \tag{9.24}$$

(see (8.7), (8.8) and (8.9)). In the critical case $2N=n$, the two conditions in (9.24) coincide, and have to be supplemented by $\dot{Q}_n^{res}(0)=Q_n$. The proof is straightforward and we omit the details. □

Finally, we outline a proof of Conjecture 9.2 for the conformally flat pseudo-spheres of Section 7. Full details are given in [JK]. First, we make the assertion more explicit. By conformal flatness, the Poincaré-Einstein metric is given by $g_+ = r^{-2}(dr^2 + g_r)$ with $g_r = g - \mathsf{P}r^2 + \mathsf{P}^2 r^4/4$ ([FG2], [J1]). Now using

$$\mathsf{P} = \frac{1}{2}\begin{pmatrix} g_{\mathbb{S}^q} & 0 \\ 0 & g_{\mathbb{S}^p} \end{pmatrix} \quad \text{and} \quad \mathsf{P}^2 = \frac{1}{4}\begin{pmatrix} g_{\mathbb{S}^q} & 0 \\ 0 & -g_{\mathbb{S}^p} \end{pmatrix}, \tag{9.25}$$

we find

$$g_+ = r^{-2}(dr^2 + (1-r^2/4)^2 g_{\mathbb{S}^q} - (1+r^2/4)^2 g_{\mathbb{S}^p}). \tag{9.26}$$

In particular, the volume function $v(r)$ is given by

$$v(r) = (1-r^2/4)^q (1+r^2/4)^p. \tag{9.27}$$

It follows that Conjecture 9.2 is equivalent to

$$1 + \sum_{N\ge 1} \frac{r^N}{N!(N-1)!}\left(\sum_{a+|J|=N} (-1)^a m_{(J,a)} P_{2J}(Q_{2a})\right) = (1-r)^{\frac{q}{2}}(1+r)^{\frac{p}{2}}.$$

But since Q-curvatures are constant, this, in turn, is equivalent to the summation formulas

$$\sum_{|I|=N} m_I \frac{P_{2I}(1)}{\frac{n}{2} - I_{\text{last}}} = N!(N-1)!\left(\sum_{M=0}^{N} (-1)^M \binom{\frac{q}{2}}{M}\binom{\frac{p}{2}}{N-M}\right), \quad N \ge 1, \tag{9.28}$$

where I_{last} denotes the last entry of the composition I.

Theorem 9.2 ([JK]). (9.28) *holds true.*

The proof of Theorem 9.2 rests on an extension of Lemma 6.2.

Corollary 9.1. *Conjecture 9.2 holds true on pseudo-spheres.*

Note that

$$g_+ = r^{-2}(dr^2 + (1-r^2/4)^2 g_{\mathbb{S}^{n-1}} - (1+r^2/4)^2 g_{\mathbb{S}^1})$$

is nothing else than the metric on the anti-de Sitter space AdS_{n+1} of dimension $n+1$. In fact, AdS_{n+1} is defined as the hyper-surface

$$C = \left\{x_1^2 + \cdots + x_{n-1}^2 - y_1^2 - y_2^2 = -1\right\} \subset \mathbb{R}^{n+2}$$

with the metric induced by $g_0 = dx_1^2 + \cdots + dx_{n-1}^2 - dy_1^2 - dy_2^2$. The map

$$\kappa : (0,1) \times \mathbb{S}^{n-1} \times \mathbb{S}^1 \ni (r,x,y) \mapsto \left(\frac{1-r^2}{2r}x, \frac{1+r^2}{2r}y\right) \in C$$

pulls back g_0 to

$$\frac{1}{r^2}\left(dr^2 + \left(1-r^2\right)^2 \frac{1}{4} g_{\mathbb{S}^{n-1}} - \left(1+r^2\right)^2 \frac{1}{4} g_{\mathbb{S}^1}\right),$$

and the substitution $r \mapsto r/2$ yields g_+.

Thus, a special case of Corollary 9.1 is a duality which relates the volume function v of anti-de Sitter space AdS_{n+1} to the generating function $\mathcal{G}$ on its boundary $\mathbb{S}^{n-1,1}$.

10. A related family of examples

In the present section, we confirm the general picture for a family of Riemannian metrics with terminating Poincaré-Einstein metrics discussed in [GoL]. The results basically follow from the corresponding results for pseudo-spheres $\mathbb{S}^{q,p}$.

We consider product manifolds $M^n = \mathbb{S}^q \times \mathbb{H}^p$, $n = q+p$ with product metrics $g_{\mathbb{S}^q} + g_{\mathbb{H}^p}$ given by the respective constant curvature metrics of curvature ± 1 on the factors. The following result describes the associated Poincaré-Einstein metrics.

Theorem 10.1 ([GoL], Theorem 4.1)**.** *The metric*

$$g_+ = r^{-2}\left(dr^2 + \left(1-r^2/4\right)^2 g_{\mathbb{S}^q} + \left(1+r^2/4\right)^2 g_{\mathbb{H}^p}\right) \tag{10.1}$$

on $(0,2)\times M^n$ *satisfies*

$$\operatorname{Ric}(g_+) = -ng_+.$$

The corresponding result in [GoL] is actually more general: the factors $\mathbb{S}^q$ and $\mathbb{H}^p$ can be replaced by Einstein spaces with suitably related scalar curvatures. The main feature of these metrics is that their Schouten tensors decompose as the sum of the respective Schouten tensors of the factors. For $\mathbb{S}^q \times \mathbb{H}^p$,

$$\mathsf{P}_{\mathbb{S}^q\times\mathbb{H}^p} = \begin{pmatrix} \mathsf{P}_{\mathbb{S}^q} & 0 \\ 0 & \mathsf{P}_{\mathbb{H}^p}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} g_{\mathbb{S}^q} & 0 \\ 0 & -g_{\mathbb{H}^p}\end{pmatrix}. \tag{10.2}$$

The metric (10.1) and the decomposition (10.2) should be compared with the metric (9.26) and the decomposition (9.25). Note that $\operatorname{tr}(\mathsf{P}) = (q-p)/2$ in both cases.

The GJMS-operators P_{2N} on M^n can be obtained by replacing $\Delta_{\mathbb{S}^p}$ by $-\Delta_{\mathbb{H}^p}$ in Theorem 7.2. In fact, by [GZ] the GJMS-operators appear in the asymptotics of the eigenfunctions of the Laplacian of the corresponding Poincaré-Einstein metric. Hence the explicit formulas (9.26) and (10.1) imply the claim. Thus, we have

Theorem 10.2. *On* $\mathbb{S}^q \times \mathbb{H}^p$*, the GJMS-operators factorize as*

$$P_{4N} = \prod_{j=1}^N \left[(B^2-C^2)^2 - 2(2j-1)^2(B^2+C^2) + (2j-1)^4\right] \tag{10.3}$$

and

$$P_{4N+2} = (-B^2+C^2)\prod_{j=1}^N \left[(B^2-C^2)^2 - 2(2j)^2(B^2+C^2) + (2j)^4\right], \tag{10.4}$$

where

$$B^2 = -\Delta_{\mathbb{S}^q} + \left(\frac{q-1}{2}\right)^2 \quad \textit{and} \quad C^2 = \Delta_{\mathbb{H}^p} + \left(\frac{p-1}{2}\right)^2. \tag{10.5}$$

In particular,

$$P_2 = -B^2 + C^2 = \Delta_{\mathbb{S}^q} + \Delta_{\mathbb{H}^p} - \left(\frac{n}{2}-1\right) Q_2$$

with

$$Q_2 = \operatorname{tr}(\mathsf{P}) = \frac{q-p}{2}$$

and

$$P_4 = (\Delta_{\mathbb{S}^q} + \Delta_{\mathbb{H}^p})^2 - \left(\frac{n}{2}-1\right)(q-p)(\Delta_{\mathbb{S}^q}+\Delta_{\mathbb{H}^p}) + 2(\Delta_{\mathbb{S}^q}-\Delta_{\mathbb{H}^p}) + \left(\frac{n}{2}-2\right) Q_4$$

with

$$Q_4 = \frac{(p+q)}{2}\frac{(q-p-2)}{2}\frac{(q-p+2)}{2}.$$

The operators P_{2N}, acting on functions which are constant on one of the factors, can be written as products of shifted Laplace operators. More precisely, we identify the restriction of P_{2N} to functions which are constant on $\mathbb{H}^p$ with an operator P_{2N}^+ on $C^\infty(\mathbb{S}^q)$. Similarly, we identify the restriction of P_{2N} to functions which are constant on $\mathbb{S}^p$ with an operator P_{2N}^- on $C^\infty(\mathbb{H}^p)$. The following product formulas generalize the product formula (2.2) on $\mathbb{S}^n$ and its analog (2.3) (for $\tau = -n(n-1)$) on $\mathbb{H}^n$.

Corollary 10.1.

$$P_{2N}^+ = \prod_{j=0}^{N-1}\left(\Delta_{\mathbb{S}^q} + \left(\frac{p+q}{2} - N + 2j\right)\left(\frac{p-q}{2} - N + 2j + 1\right)\right) \tag{10.6}$$

and

$$P_{2N}^- = \prod_{j=0}^{N-1}\left(\Delta_{\mathbb{H}^p} + \left(\frac{p+q}{2} - N + 2j\right)\left(\frac{p-q}{2} + N - 2j - 1\right)\right). \tag{10.7}$$

Proof. For the operators P_{4N}^+, we observe that the product of the factors for $j = k$ and $2N-1-k$ in (10.6) coincides with the factor for $j = N-k$ in (10.3). In (10.6) for P_{4N+2}^+, the factor for $j = N$ is P_2^+. The remaining $2N$ factors group together similarly as before. Analogous arguments apply for the operators P_{2N}^-. □

Corollary 10.1 has some consequences for the kernel of the critical GJMS-operator P_n on $\mathbb{S}^q \times \mathbb{H}^p$, $q+p=n$. By the conformal covariance of P_n, this space is an invariant of the conformal class of the metric. On functions which are constant on $\mathbb{H}^p$, P_n reduces to

$$\prod_{j=0}^{\frac{n}{2}-1}(\Delta_{\mathbb{S}^q} + 2j(2j+1-q)).$$

The latter formula shows that the $SO(q+1)$-modules

$$\ker(\Delta_{\mathbb{S}^q} + L(L+q-1)) = \ker(\Delta_{\mathbb{S}^q} + 2j(2j+1-q)),$$

where

$$L = q-1-2k \quad \text{and} \quad j = q-1-k \quad \text{with} \quad k = 0, \cdots, \lfloor (q-1)/2 \rfloor,$$

of spherical harmonics of degree L induce subspaces of $\ker(P_n)$. This generalizes an observation of M. Eastwood and M. Singer for P_4 on $\mathbb{S}^2 \times \mathbb{H}^2$. Similarly, on functions which are constant on $\mathbb{S}^q$, P_n reduces to

$$\prod_{j=0}^{\frac{n}{2}-1}(\Delta_{\mathbb{H}^p} + 2j(p-1-2j)).$$

Hence the non-trivial $SO(1,p)$-modules

$$\ker(\Delta_{\mathbb{H}^p} + 2j(p-1-2j)), \; j = 0, \dots, \lfloor (p-1)/2 \rfloor,$$

induce subspaces of $\ker(P_n)$.

These observations extend to the critical GJMS-operator P_n on compact spaces of the form $\mathbb{S}^q \times \Gamma\backslash\mathbb{H}^p$, where the discrete subgroup $\Gamma \subset SO(1,p)$ operates with a compact quotient on $\mathbb{H}^p$.

The meaning of a non-trivial kernel of P_n for the Q-curvature prescription problem on compact manifolds was discussed in [Go2].

The following result should be compared with Corollary 7.1.

Corollary 10.2. *On $\mathbb{S}^q \times \mathbb{H}^p$,*

$$Q_{2N} = \prod_{j=1}^{N-1} \left(\frac{p+q}{2} + N - 2j \right) \prod_{j=0}^{N-1} \left(\frac{q-p}{2} - N + 1 + 2j \right), \; N \geq 1.$$

Corollary 10.2 implies that the critical Q-curvature Q_n, $n = q+p$, of $\mathbb{S}^q \times \mathbb{H}^p$ equals

$$Q_n = \prod_{j=1}^{\frac{n}{2}-1} (n-2j) \prod_{j=0}^{\frac{n}{2}-1} (2j+1-p).$$

This yields

Corollary 10.3. *For odd q and p, the critical Q-curvature of $\mathbb{S}^q \times \mathbb{H}^p$ vanishes.*

Alternatively, this result follows from the holographic formula [GJ]. In fact, by Remark 6.16.1 in [J1], Q_n is a constant multiple of the coefficient of r^n in

$$\left(1 - \frac{r^2}{2}\right)^q \left(1 + \frac{r^2}{2}\right)^p.$$

Hence it suffices to prove that

$$\sum_{i=0}^{q} (-1)^i \binom{q}{i} \binom{n-q}{\frac{n}{2}-i} = 0$$

for odd q. But this sum equals

$$\frac{1}{2} \left(\sum_{i=0}^{q} (-1)^i \binom{q}{i} \binom{n-q}{\frac{n}{2}-i} + (-1)^q \sum_{i=0}^{q} (-1)^i \binom{q}{q-i} \binom{n-q}{\frac{n}{2}-(q-i)} \right).$$

Hence it vanishes for odd q.

Corollary 10.3 also appears in [C].

Now direct calculations yield the recursive formulas

$$P_4 = (P_2^2)^0 + 2(\Delta_{\mathbb{S}^q} - \Delta_{\mathbb{H}^p}) + \left(\frac{n}{2} - 2\right) Q_4$$

and

$$P_6 = \left(2P_2P_4 + 2P_4P_2 - 3P_2^3\right)^0 + 12(\Delta_{\mathbb{S}^q} + \Delta_{\mathbb{H}^p}) - \left(\frac{n}{2} - 3\right) Q_6$$

with

$$Q_6 = \frac{(p+q-2)}{2}\frac{(p+q+2)}{2}\frac{(q-p-4)}{2}\frac{(q-p)}{2}\frac{(q-p+4)}{2}.$$

These are special cases of Theorem 4.1 and Theorem 4.2, respectively.

More generally, the following summation formula is a Riemannian analog of Theorem 7.3.

Theorem 10.3. *On $\mathbb{S}^q \times \mathbb{H}^p$,*

$$\mathcal{M}_{4N} = (2N)!(2N-1)!\left(\frac{1}{2}-B^2-C^2\right), \ N \geq 1$$

and

$$\mathcal{M}_{4N+2} = (2N+1)!(2N)!\,P_2, \ N \geq 0$$

with B^2 and C^2 as defined in (10.5).

In particular, we find

$$\mathcal{M}^0_{2N} = -2^{N-1}N!(N-1)!\,\delta(\mathsf{P}^{N-1}\#d)$$

using (10.2). This result is a special case of Conjecture 4.1. The volume function v of g_+ is given by

$$v(r) = \left(1-r^2/4\right)^q\left(1+r^2/4\right)^p$$

and the relation

$$\mathcal{G}(r^2/4) = \sqrt{v(r)}$$

follows from the corresponding relation for the pseudo-spheres.

11. Extension beyond conformally flat metrics

In the present section, we formulate an extension of Conjecture 4.1 to general metrics, and discuss that extension for $N = 4$ in the critical dimension $n = 8$. In this case, the formulation involves the first two of Graham's extended obstruction tensors [G4].

In order to motivate the following formulations, we note that (4.4) for general metrics differs from the formula for locally conformally flat metrics only by the second-order operator

$$-\frac{16}{n-4}\delta(\mathcal{B}\#d). \tag{11.1}$$

Moreover, (4.6) and (4.7) show that P_6, when viewed as a rational function in n, has a simple pole at $n = 4$ with residue

$$\mathcal{R}_6 = -16(\delta(\mathcal{B}\#d) - (\mathcal{B},\mathsf{P})).$$

A direct proof shows that $\mathcal{R}_6$ is conformally covariant, i.e.,

$$e^{5\varphi}\circ\hat{\mathcal{R}}_6 = \mathcal{R}_6\circ e^{-\varphi}.$$

Here $\hat{\mathcal{R}}_6$ denotes $\mathcal{R}_6$ for the metric $\hat{g} = e^{2\varphi}g$. The same convention will be used in the following for other tensors. The operator $\mathcal{R}_6$ obstructs the existence of P_6 for metrics with $\mathsf{C} \neq 0$ in dimension $n = 4$.

We interpret (11.1) as

$$2!2^3\delta(\Omega^{(1)}\#d) \tag{11.2}$$

using the first extended obstruction tensor

$$\Omega^{(1)} = \frac{\mathcal{B}}{4-n}. \tag{11.3}$$

Similarly, for general metrics, the formula for $\mathbf{P}_8$ in Conjecture 5.1 should contain the additional term

$$3!2^4\delta(\Omega^{(2)}\#d), \tag{11.4}$$

where $\Omega^{(2)}$ is the second extended obstruction tensor [G4].

The first two extended obstruction tensors are defined by

$$\Omega^{(1)}_{ij} = \tilde{R}_{\infty ij\infty}|_{\rho=0,t=1} \quad \text{and} \quad \Omega^{(2)}_{ij} = \tilde{\nabla}_\infty(\tilde{R})_{\infty ij\infty}|_{\rho=0,t=1}, \tag{11.5}$$

where $\tilde{R}$ denotes the curvature tensor of the Fefferman-Graham ambient metric. The extended obstruction tensors $\Omega^{(k)}$ are special conformal curvature tensors (in the sense of [FG2]). In particular, they vanish if $\mathsf{C} = 0$, and their conformal variations only depend on first-order derivatives of φ.

The tensors $\Omega^{(k)}$ can be regarded as rational functions in the dimension n. The Schouten tensor P and the extended obstruction tensors play the role of canonical building blocks of the ambient metric and hence also of derived quantities such as the holographic coefficients. For the details we refer to [G4].

The following result proves the conformal covariance of a generalization of $\mathbf{P}_8$ (Theorem 5.1) in the critical dimension.

Theorem 11.1. *On manifolds of dimension $n = 8$, the self-adjoint operator*

$$\mathbf{P}_8 \stackrel{def}{=} \mathcal{P}^0_8 - 3!2^4\delta(\left[\Omega^{(2)} - 4(\mathsf{P}\Omega^{(1)} + \Omega^{(1)}\mathsf{P}) + 12\mathsf{P}^3\right]\#d) \tag{11.6}$$

is conformally covariant, i.e.,

$$e^{8\varphi}\mathbf{P}_8(e^{2\varphi}g) = \mathbf{P}_8(g) \tag{11.7}$$

for all metrics g and all $\varphi \in C^\infty(M)$.

Although the definition of $\mathbf{P}_8$ involves the first two extended obstruction tensors, the following proof does not depend on the explicit formulas [G4] for these tensors in terms of P, C, $\mathcal{C}$ and $\mathcal{B}$. It is natural to ask whether $\mathbf{P}_8$ coincides with P_8. As already noted in Section 5, it seems natural to approach this problem by conformal variation of Q_8 on the basis of the recursive formula (9.6) and the explicit expression (11.16) for v_8 derived in [G4].

Proof. It suffices to prove the infinitesimal conformal covariance. By Theorem 3.1,

$$(d/dt)|_0\left(e^{8t\varphi}\mathcal{P}^0_8(e^{2t\varphi}g)\right) = 9[\mathcal{M}_6,[\mathcal{M}_2,\varphi]]^0 + 18[\mathcal{M}_4,[\mathcal{M}_4,\varphi]]^0 + 3[\mathcal{M}_2,[\mathcal{M}_6,\varphi]]^0.$$

Since $\mathcal{M}_2$, $\mathcal{M}_4$ and $\mathcal{M}_6$ are second-order operators,

$$(d/dt)|_0\left(e^{8t\varphi}\mathcal{P}^0_8(e^{2t\varphi}g)\right) = 9[\mathcal{M}^0_6,[\mathcal{M}^0_2,\varphi]]^0 + 18[\mathcal{M}^0_4,[\mathcal{M}^0_4,\varphi]]^0 + 3[\mathcal{M}^0_2,[\mathcal{M}^0_6,\varphi]]^0.$$

The relations

$$\mathcal{M}_6^0 = -48\mathcal{T}_2 - \frac{16}{n-4}\delta(\mathcal{B}\#d), \quad \mathcal{M}_4^0 = -4\mathcal{T}_1 \quad \text{and} \quad \mathcal{M}_2^0 = \Delta$$

imply that the above sum coincides with

$$144\left(-3[\mathcal{T}_2,[\Delta,\varphi]]^0 + 2[\mathcal{T}_1,[\mathcal{T}_1,\varphi]]^0 - [\Delta,[\mathcal{T}_2,\varphi]]^0\right) - \frac{16}{n-4}\left(9[\delta(\mathcal{B}\#d),[\Delta,\varphi]]^0 + 3[\Delta,[\delta(\mathcal{B}\#d),\varphi]]^0\right). \tag{11.8}$$

By the same arguments as in the conformally flat case, it only remains to prove that the sum of

$$144\left(8\mathcal{C}_{rij}\mathsf{P}_k^r + 4\mathcal{C}_{krj}\mathsf{P}_i^r + 4\mathcal{C}_{kir}\mathsf{P}_j^r\right)\varphi^i \operatorname{Hess}^{jk}(u) \tag{11.9}$$

(see (3.8)) and the main part of

$$-\frac{16}{n-4}\left(9[\delta(\mathcal{B}\#d),[\Delta,\varphi]]^0 + 3[\Delta,[\delta(\mathcal{B}\#d),\varphi]]^0\right) \tag{11.10}$$

coincides with

$96 \times$ the main part of the conformal variation of $\delta([\Omega^{(2)} - 4(\mathsf{P}\Omega^{(1)} + \Omega^{(1)}\mathsf{P})]\#d)$.

Now (11.10) has the main part

$$-\frac{16}{n-4}\left[(18\nabla_i(\mathcal{B})_{jk} - 12\nabla_k(\mathcal{B})_{ij})\varphi^i - 48\mathcal{B}_{ij}\operatorname{Hess}_k^i(\varphi)\right]\operatorname{Hess}^{jk}(u). \tag{11.11}$$

By the transformation law of the Bach tensor (see (13.16)),

$$e^{2\varphi}\hat{\Omega}_{ij}^{(1)} = \Omega_{ij}^{(1)} - (\mathcal{C}_{kij} + \mathcal{C}_{kji})\varphi^k - \mathsf{C}_{kijl}\varphi^k\varphi^l.$$

Hence the main part of the conformal variation of $\frac{1}{2}\delta((\mathsf{P}\Omega^{(1)} + \Omega^{(1)}\mathsf{P})\#d)$ is

$$\left[\Omega_{ir}^{(1)}\operatorname{Hess}_j^r(\varphi) + (\mathcal{C}_{lir} + \mathcal{C}_{lri})\mathsf{P}_j^r\varphi^l\right]\operatorname{Hess}^{ij}(u). \tag{11.12}$$

In order to determine the conformal variation of $\Omega^{(2)}$, we apply some results of [G4]. Under conformal changes,

$$e^{4\varphi}\hat{\Omega}_{ij}^{(2)} = \Omega_{ij}^{(2)} - \mathcal{C}_{ijl}^{(2)}\varphi^l + O(|\nabla\varphi|^2)$$

(Proposition 2.7) with the second Cotton tensor $\mathcal{C}^{(2)}$ (Definition 2.4).[2] Now the relation

$$\mathcal{C}_{ijk}^{(2)} = \left(3\tilde{\nabla}_k(\tilde{R})_{\infty ij\infty} - \tilde{\nabla}_j(\tilde{R})_{\infty li\infty} - \tilde{\nabla}_i(\tilde{R})_{\infty lj\infty}\right)\big|_{\rho=0,t=1}$$

and the formula

$$\tilde{\nabla}_l(\tilde{R})_{\infty ij\infty}|_{\rho=0,t=1} = \frac{\nabla_l(\mathcal{B})_{ij}}{4-n} - (\mathcal{C}_{rij} + \mathcal{C}_{rji})\mathsf{P}_l^r$$

for the covariant derivatives of $\tilde{R}$ suffice to determine the main part of the conformal variation of $\delta(\Omega^{(2)}\#d)$. It is given by the sum of

[2] We retain the convention of [G4] concerning the higher Cotton tensor: $\mathcal{C}^{(2)}$ is symmetric in the first two arguments. On the other hand, $\mathcal{C}$ is anti-symmetric in the first two arguments.

$$(11.13)\quad \frac{1}{4-n}\left(3\nabla_l(\mathcal{B})_{ij} - \nabla_j(\mathcal{B})_{li} - \nabla_i(\mathcal{B})_{lj}\right)\varphi^l\,\mathrm{Hess}^{ij}(u) = \frac{1}{4-n}\left(3\nabla_l(\mathcal{B})_{ij} - 2\nabla_j(\mathcal{B})_{li}\right)\varphi^l\,\mathrm{Hess}^{ij}(u)$$

and

$$\left(-3(\mathcal{C}_{rij}+\mathcal{C}_{rji})\mathsf{P}^r_l + 2(\mathcal{C}_{ril}+\mathcal{C}_{rli})\mathsf{P}^r_j\right)\varphi^l\,\mathrm{Hess}^{ij}(u).$$

Now $96 \times (11.13)$ coincides with the terms in (11.11) which contain a derivative of $\mathcal{B}$. Next, the relation

$$96\cdot(-8)\Omega^{(1)}_{ir}\,\mathrm{Hess}^r_j(\varphi)\,\mathrm{Hess}^{ij}(u) = \frac{16}{n-4}48\mathcal{B}_{ij}\,\mathrm{Hess}^i_k(\varphi)\,\mathrm{Hess}^{jk}(u)$$

verifies the assertion for the contributions which contain two derivatives of φ. It only remains to prove that

$$(11.14)\quad 144\left(8\mathcal{C}_{rij}\mathsf{P}^r_k + 4\mathcal{C}_{krj}\mathsf{P}^r_i + 4\mathcal{C}_{kir}\mathsf{P}^r_j\right)\varphi^i\,\mathrm{Hess}^{jk}(u) = 144\left(8\mathcal{C}_{rli}\mathsf{P}^r_j + 4\mathcal{C}_{jri}\mathsf{P}^r_l + 4\mathcal{C}_{jlr}\mathsf{P}^r_i\right)\varphi^l\,\mathrm{Hess}^{ij}(u)$$

coincides with

$$(11.15)\quad 96\left[-6\mathcal{C}_{rji}\mathsf{P}^r_l + 2(\mathcal{C}_{ril}+\mathcal{C}_{rli})\mathsf{P}^r_j - 8(\mathcal{C}_{lir}+\mathcal{C}_{lri})\mathsf{P}^r_j\right]\varphi^l\,\mathrm{Hess}^{ij}(u).$$

We observe that $144\cdot 4\mathcal{C}_{jri} = -96\cdot 6\mathcal{C}_{rji}$. Next, for the last two products in (11.15) we find

$$\begin{aligned}&\big[2\nabla_r(\mathsf{P})_{il} - 2\nabla_i(\mathsf{P})_{rl} + 2\nabla_r(\mathsf{P})_{li} - 2\nabla_l(\mathsf{P})_{ri}\\ &\quad - 8\nabla_l(\mathsf{P})_{ir} + 8\nabla_i(\mathsf{P})_{lr} - 8\nabla_l(\mathsf{P})_{ri} + 8\nabla_r(\mathsf{P})_{li}\big]\mathsf{P}^r_j\\ &\quad = \left(12\nabla_r(\mathsf{P})_{il} + 6\nabla_i(\mathsf{P})_{rl} - 18\nabla_l(\mathsf{P})_{ri}\right)\mathsf{P}^r_j.\end{aligned}$$

On the other hand, the first and third term on the right-hand side of (11.14) yield

$$(8\nabla_r(\mathsf{P})_{li} + 4\nabla_i(\mathsf{P})_{lr} - 12\nabla_l(\mathsf{P})_{ir})\mathsf{P}^r_j\varphi^l\,\mathrm{Hess}^{ij}(u)$$

using the symmetry of $\mathrm{Hess}^{ij}(u)$. Now the obvious relation

$$96(12,6,-18) = 144(8,4,-12)$$

completes the proof. □

Next, we describe a second motivation of the $\Omega^{(2)}$-term in (11.6) in general dimensions. For this we recall that P_8^0 is determined by conformal variation of Q_8. The difference

$$Q_8 - 3!4!2^7 v_8$$

can be expressed in various ways in terms of lower order constructions. Here one either applies a generalization of the holographic formula (9.3) or uses a generalization of the recursive formula (9.6) to subcritical cases. Now Graham [G4] has shown that[3]

$$(11.16)\quad 2^4 v_8 = \mathrm{tr}(\wedge^4\mathsf{P})$$

[3] Here we correct a misprint in the last term of formula (2.23) in [G4]

$$+\frac{1}{3}\operatorname{tr}(\mathsf{P}^2\Omega^{(1)})-\frac{1}{3}\operatorname{tr}(\mathsf{P})\operatorname{tr}(\mathsf{P}\Omega^{(1)})-\frac{1}{12}\operatorname{tr}(\mathsf{P}\Omega^{(2)})-\frac{1}{12}\operatorname{tr}(\Omega^{(1)}\Omega^{(1)}).$$

In particular, Q_8 contains the contribution

$$-96(\Omega^{(2)},\mathsf{P}).$$

Now conformal variation yields

$$96(\Omega^{(2)},\operatorname{Hess}(\varphi)),$$

up to a first order operator. This motivates the $\Omega^{(2)}$-term in (11.6).

A similar argument motivates the $(\mathsf{P}\Omega^{(1)}+\Omega^{(1)}\mathsf{P})$-term. In fact, by (11.16), the quantity $(\mathsf{P}^2,\Omega^{(1)})$ contributes to Q_8 with the coefficient $3!2^6$. Now conformal variation of $(\mathsf{P}^2,\Omega^{(1)})$ yields $-2(\mathsf{P}\Omega^{(1)},\operatorname{Hess}(\varphi))+\cdots$. This motivates the contribution

$$3!2^6\delta((\mathsf{P}\Omega^{(1)}+\Omega^{(1)}\mathsf{P})\#d)$$

in (11.6).

Now universality claims that, in all dimensions $n\ge 8$, the analogous operator

$$\mathcal{P}_8^0-3!2^4\delta(\left[\Omega^{(2)}-4(\mathsf{P}\Omega^{(1)}+\Omega^{(1)}\mathsf{P})+12\mathsf{P}^3\right]\#d)+\left(\frac{n}{2}-4\right)Q_8$$

is conformally covariant, too (and coincides with P_8). This extends Conjecture 5.1. In particular, regarding the operator as a rational function in n, it has a simple pole at $n=6$. For the residue we find

$$-96\delta(\operatorname{Res}_{n=6}(\Omega^{(2)})\#d)+96(\operatorname{Res}_{n=6}(\Omega^{(2)}),\mathsf{P})=-48(\delta(\mathcal{O}\#d)-(\mathcal{O},\mathsf{P}))$$

by the residue formula ([G4], Proposition 2.8)

$$2\operatorname{Res}_{n=6}(\Omega^{(2)})=\mathcal{O}.$$

Here $\mathcal{O}$ denotes the Fefferman-Graham obstruction tensor in dimension six. An explicit formula for $\mathcal{O}$ can be found, for instance, in [GH]. As the Bach tensor $\mathcal{B}$ in dimension 4, $\mathcal{O}$ is trace-free and divergence-free. Moreover, its transformation law $e^{4\varphi}\hat{\mathcal{O}}=\mathcal{O}$ in dimension six generalizes $e^{2\varphi}\hat{\mathcal{B}}=\mathcal{B}$ in dimension 4. A direct calculation, using these properties, confirms that the operator

$$\mathcal{R}_8=\delta(\mathcal{O}\#d)-(\mathcal{O},\mathsf{P})=-(\mathcal{O},\operatorname{Hess})-(\mathcal{O},\mathsf{P})$$

is conformally covariant (in dimension six), i.e.,

$$e^{7\varphi}\circ\hat{\mathcal{R}}_8=\mathcal{R}_8\circ e^{-\varphi}.$$

It obstructs the existence of $\mathbf{P}_8$ for general metrics in dimension six.

Similarly, in dimension $n=4$, the existence of $\mathbf{P}_8$ is obstructed by a conformally covariant self-adjoint differential operator of order four with main part $(\mathcal{B},\operatorname{Hess}\Delta)$.

Finally, we show that the above results can be seen as special cases of the following extension of Conjecture 4.1.

Conjecture 11.1 **(Universal recursive formulas for GJMS-operators).** *Let the integer $N \ge 1$ satisfy $2N \le n$ if n is even. Then on any Riemannian manifold of dimension $n \ge 3$, the GJMS-operator P_{2N} is given by the recursive formula*

$$P_{2N} = \mathcal{P}^0_{2N} - a_N \delta(D_{2N-2}\#d) + (-1)^N \left(\frac{n}{2} - N\right) Q_{2N}. \tag{11.17}$$

Here $a_N = (2^{N-1}(N-1)!)^2$, and the natural symmetric bilinear forms $D_2(g), D_4(g), \dots$ are the coefficients of the Taylor series

$$g_r^{-1} = g^{-1} + r^2 D_2(g) + r^4 D_4(g) + \cdots$$

of the inverse of the symmetric bilinear form g_r so that $r^{-2}(dr^2 + g_r)$ is the Poincaré-Einstein metric associated to g.

We recall that in the locally conformally flat case, $g_r = (1 - r^2/2\mathsf{P})^2$. Hence

$$g_r^{-1} = \sum_{r \ge 1} N \mathsf{P}^{N-1} (r^2/2)^{N-1}.$$

It follows that

$$a_N D_{2N-2} = 2^{N-1} N!(N-1)! \mathsf{P}^{N-1}.$$

This proves that Conjecture 11.1 extends Conjecture 4.1.

In order to see that (11.17) also extends the formulas (4.4) for P_6 and (11.6) for P_8, we use the fact that for general metrics the coefficients in the Taylor series

$$g_r = \sum_{N \ge 0} \left(-\frac{r^2}{2}\right)^N \frac{1}{N!} g_{(2N)} \stackrel{\text{def}}{=} \sum_{N \ge 0} d_{2N} r^{2N}$$

are given by

$$\begin{aligned} \frac{1}{2} g_{(2)} &= \mathsf{P}, \\ \frac{1}{2} g_{(4)} &= \Omega^{(1)} + \mathsf{P}^2, \\ \frac{1}{2} g_{(6)} &= \Omega^{(2)} + 2(\mathsf{P}\Omega^{(1)} + \Omega^{(1)}\mathsf{P}) \end{aligned}$$

(see [G4], (2.22)). Now we have

$$g_r^{-1} = g^{-1} + D_2 r^2 + D_4 r^4 + D_6 r^6 + \cdots$$

with

$$D_2 = -d_2, \quad D_4 = -d_4 + d_2^2, \quad D_6 = -d_6 + (d_2 d_4 + d_4 d_2) - d_2^3.$$

Hence

$$\begin{aligned} D_2 &= \frac{1}{2} g_{(2)} = \mathsf{P}, \\ D_4 &= -\frac{1}{8} g_{(4)} + \frac{1}{4} g_{(2)}^2 = \frac{1}{4}(-\Omega^{(1)} + 3\mathsf{P}^2) \end{aligned}$$

and

$$D_6 = \frac{1}{48} g_{(6)} - \frac{1}{16}(g_{(2)} g_{(4)} + g_{(4)} g_{(2)}) + \frac{1}{8} g_{(2)}^3 = \frac{1}{24}(\Omega^{(2)} - 4(\Omega^{(1)}\mathsf{P} + \mathsf{P}\Omega^{(1)}) + 12\mathsf{P}^3).$$

This proves the claim.

It is natural to summarize the assertions of Conjecture 11.1 for the non-constant terms in form of the identity

$$\sum_{N\geq 1} \frac{\mathcal{M}^0_{2N}(g)}{(N-1)!(N-1)!} \left(\frac{r^2}{4}\right)^{N-1} = -\delta(g_r^{-1}\#d). \tag{11.18}$$

The latter relation extends (4.2). Of course, in even dimensions, (11.18) is to be understood as an identity of finite series.

By [G4], the symmetric bilinear forms D_{2N} are given by *universal* formula in terms of extended obstruction tensors and P. Combined with Graham's [G4] universal formulas for the holographic coefficients in terms of extended obstruction tensors and the recursive formulas for Q-curvatures (Conjecture 9.1), Conjecture 11.1 yields universal formulas for all GJMS-operators in terms of the building blocks $\mathsf{P}, \Omega^{(1)}, \Omega^{(2)}, \ldots$.

12. Further comments and open problems

In the locally conformally flat case, Theorem 3.2 deduces the conformal covariance of the critical operator

$$\mathbf{P}_n = \mathcal{P}^0_n - c_{\frac{n}{2}}\mathcal{T}_{\frac{n}{2}-1} = \Delta^{\frac{n}{2}} + LOT$$

on a manifold of even dimension n from the relations

$$\mathcal{M}^0_{2N} = -c_N\mathcal{T}_{N-1},\ 2N < n \tag{12.1}$$

for all subcritical GJMS-operators. Theorem 5.1 provides an application to a construction of a conformally covariant fourth power $\mathbf{P}_8$ of Δ in dimension 8. It rests on the fact that, in this case, the assumptions in Theorem 3.2, i.e., (12.1) for $N = 2$ and $N = 3$, are known to be satisfied by Theorem 4.1 and Theorem 4.2.

The identification of $\mathbf{P}_8$ with P_8, however, remains open. As already mentioned in Section 5, an approach through conformal variation of Q_8 based on (9.6) seems feasible. In this connection, the universality of (9.6) would play a crucial role.

In an analogous proof of the conformal covariance of

$$\mathbf{P}_{10} = \mathcal{P}^0_{10} - c_5\mathcal{T}_4$$

in dimension $n = 10$, the only missing piece is a substitute of the relation (12.1) for $N = 4$ in dimension $n = 10$. The principle of universality predicts that this relation actually holds true in all higher dimensions.

A direct identification of the computer-derived formula [GoP] for Q_8 (in general dimension) with the universal recursive formula (9.6) is a challenging task. We illustrate the issue by relating the respective coefficients of $(\Delta^2(\mathsf{P}), \mathsf{P})$ and J^4 in both formulas. In the extension of (9.6) to general dimensions, the contribution $(\Delta^2(\mathsf{P}), \mathsf{P})$ appears through

$$-3P_2(Q_6), \quad (9P_4 - 12P_2^2)(Q_4) \quad \text{and} \quad 3!4!2^7 v_8. \tag{12.2}$$

Now the contribution

$$-2P_2(Q_4) + \frac{16}{n-4}(\mathcal{B}, \mathsf{P})$$

in Q_6 (see (4.6), (4.7)) yields

$$-24\frac{n-2}{n-4}(\Delta^2\mathsf{P},\mathsf{P}).$$

The second term in (12.2) gives $12(\Delta^2(\mathsf{P}),\mathsf{P})$. Finally, by (11.16) and

$$(n-4)(n-6)\Omega^{(2)} = \Delta^2(\mathsf{P}) + \cdots$$

(see [G4]), the last term in (12.2) gives

$$-\frac{96}{(n-4)(n-6)}(\Delta^2(\mathsf{P}),\mathsf{P}).$$

Adding these results, reproduces the coefficient

$$-12\frac{n-2}{n-6}$$

in [GoP]. Next, we apply (9.6) to determine the coefficient of J^4. A calculation using

$$Q_2 = \mathsf{J}, \quad Q_4 = \frac{n}{2}\mathsf{J}^2 + \cdots \text{ (by (1.3))}$$

and

$$Q_6 = \frac{(n-2)(n+2)}{4}\mathsf{J}^3 + \cdots \text{ (by (4.6), (4.7))}$$

yields

$$Q_8 = \frac{(n-4)n(n+4)}{8}\mathsf{J}^4 + \cdots .$$

Note that in this calculation all terms in (9.6) contribute in a non-trivial way. These results for Q_6 and Q_8 fit with [GoP]. More generally, the recursive formula predicts the contribution

$$\prod_{j=1}^{N-1}\left(\frac{n}{2}-N+2j\right)\mathsf{J}^N$$

in Q_{2N}, which, in the critical case, is only caused by the v_n-term.

In general, (9.8) identifies Q_{2N} with a sum of the form

$$\mathcal{Q}_{2N} + \cdots + (-1)^N N!(N-1)!2^{2N-1}v_{2N}. \tag{12.3}$$

Now infinitesimal conformal variation of this sum yields an operator of the form $\mathcal{P}_{2N} + \cdots$. Thus, Conjecture 9.1 implies a representation formula $P_{2N} = \mathcal{P}_{2N} + \cdots$. Conjecture 4.1 actually predicts huge cancellations in that sum. It is crucial to understand the mechanism of these cancellations.

A good understanding of the infinitesimal conformal variation of the quantity in (12.3) would be an important ingredient in a proof of the conformal covariance of the subcritical operator

$$\begin{aligned}\mathcal{P}^0_{2N} &- c_N\delta(\mathsf{P}^{N-1}\#d)\\ &+\left(\frac{n}{2}-N\right)(-1)^N(\mathcal{Q}_{2N} + \cdots + (-1)^N N!(N-1)!2^{2N-1}v_{2N}),\ 2N<n\end{aligned} \tag{12.4}$$

(in the locally conformally flat case) along similar lines as in the critical case. The constant term of the operator (12.4) is given by the sum in (12.3). Such a proof is independent of the recognition of the constant term as Q_{2N}. We shall illustrate this for $N = 2$ in Section 13.1.

The proofs of the conformal covariance of the respective critical operators P_4, P_6 and $\mathbf{P}_8$ rest only on the conformal transformation properties of the lower order GJMS-operators in the respective primary parts, and of the tensors which contribute to the secondary parts. In particular, the proof of Theorem 11.1 does not require explicit formulas for the extended obstruction tensors. This feature of the proofs nurtures the hope for a similar treatment of the general case.

The original ambient metric construction [GJMS] generates the operator P_{2N} from the action of the power $\Delta_{\tilde{g}}^N$ of the Laplacian of the ambient metric $\tilde{g}$ on a space of homogenous functions of a certain degree depending on N. Since the functional spaces depend on N, even the very existence of recursive formulas for these operators remains obscure from this perspective.

The method of recursive constructions of conformally covariant powers of the Laplacian described here does *not* rest on the ambient metric. Nevertheless, the construction somehow forces the Taylor coefficients of the ambient metric to appear. In fact, (11.18) states that the Taylor coefficients of the second-order operator on the right-hand side provide appropriate correction terms which can be used to make the respective operators $\mathcal{P}_{2N}^0$ conformally covariant. These correction terms contain the full information on the ambient metric. It seems that there are no alternative choices for these terms.

It is often fruitful to think of conformally covariant differential operators on general manifolds as "curved analogs" of their special cases on spheres. In the opposite direction, tractor calculus offers constructions of curved analogs of differential operators on spheres which are equivariant with respect to the conformal group. A recent manifestation of this line of thinking is the method of curved Casimir operators [CGS]. Although the present paper does not emphasize the perspective of curved analogs, it is tempting to ask for a representation theoretical interpretation of the operators $\mathcal{M}_{2N}$. In particular, regarding the identities in Theorem 6.1 and Theorem 7.3 as non-linear relations among intertwining operators for principal series representations (see (7.7)) of $O(q+1, p+1)$ motivates to ask for representation theoretical proofs of these relations.

One consequence of the holographic formula (9.3) for the critical Q-curvature Q_n is the proportionality [GZ]

$$2 \int_{M^n} Q_n vol = 2^n (-1)^{\frac{n}{2}} \left(\frac{n}{2}\right)! \left(\frac{n}{2}-1\right)! \int_{M^n} v_n vol. \tag{12.5}$$

On the other hand, (9.8) predicts that

$$2Q_n - 2^n (-1)^{\frac{n}{2}} \left(\frac{n}{2}\right)! \left(\frac{n}{2}-1\right)! v_n = 2\mathcal{Q}_n - \sum_{j=1}^{\frac{n}{2}-1} \frac{j(\frac{n}{2}-j)}{\frac{n}{2}} \binom{\frac{n}{2}}{j}^2 \Lambda_{n-2j}\Lambda_{2j}, \tag{12.6}$$

where

$$\mathcal{Q}_n = (-1)^{\frac{n}{2}-1} \sum_{|I|+a=\frac{n}{2},\, a\neq\frac{n}{2}} (-1)^a m_{(I,a)} P_{2I}(Q_{2a}).$$

Hence a proof of the identity

$$\int_{M^n} \left[2\mathcal{Q}_n - \sum_{j=1}^{\frac{n}{2}-1} \frac{j(n-2j)}{n} \binom{\frac{n}{2}}{j}^2 \Lambda_{n-2j}\Lambda_{2j} \right] vol = 0 \tag{12.7}$$

could be regarded as support for (12.6). It would be interesting to give a *direct* proof that the integrand in (12.7) is a total divergence.

In small dimensions, this can be verified directly. For instance, a calculation in dimension $n = 6$ shows that the integrand has the form

$$-2P_2^0(Q_4) + 2P_4^0(Q_2) - 3P_2^0 P_2(Q_2).$$

Similarly, we find that the *critical* Q_8 is given by the sum of the reduced primary part

$$\begin{aligned}[-3P_2^0(Q_6) - 3P_6^0(Q_2) + 9P_4^0(Q_4) \\ + 8P_2^0P_4(Q_2) - 12P_2^0P_2(Q_4) + 12P_4^0P_2(Q_2) - 18P_2^0P_2^2(Q_2)],\end{aligned}$$

the additional terms

$$12[\Delta(Q_4)Q_2 - Q_4\Delta(Q_2)] + 18[\Delta^2(Q_2) - \Delta(Q_2)^2] + 54[\Delta(Q_2)Q_2^2 - Q_2\Delta(Q_2^2)]$$

and $3!4!2^7 v_8$.

In particular, such a proof requires to verify that, for any non-trivial composition I of size $\frac{n}{2}$, the coefficient of Q_{2I} in the integrand vanishes. We confirm the vanishing in two interesting special cases. First, let $I = (p,q)$ with $|I| = \frac{n}{2}$. The coefficient of Q_{2I} in the integrand in (12.7) is given by the difference of

$$-2\left((n/2-p)m_{(p,q)} + (n/2-q)m_{(q,p)}\right)$$

and

$$\frac{p(\frac{n}{2}-p)}{\frac{n}{2}}\binom{\frac{n}{2}}{p}^2 + \frac{q(\frac{n}{2}-q)}{\frac{n}{2}}\binom{\frac{n}{2}}{q}^2.$$

Now the formula

$$m_{(p,q)} = -\binom{\frac{n}{2}-1}{p}\binom{\frac{n}{2}-1}{q}$$

(see (2.9)) shows that the difference vanishes. Next, let $I = (i,j,k)$ with $|I| = \frac{n}{2}$. (2.16) and Corollary 2.1 imply

$$\frac{m_{(i,j,k)}}{|I|-k} = -\binom{|I|}{k}^2 \frac{k(|I|-k)}{|I|} \frac{m_{(j,i)}}{|I|-i}. \tag{12.8}$$

The coefficient of Q_{2I} in the integrand in (12.7) is the sum of

$$-2\left(\frac{n}{2}-i\right)\left(\frac{n}{2}-j\right)\left(\frac{n}{2}-k\right)\sum_{\sigma\in S_3} \frac{m_{\sigma I}}{\frac{n}{2}-(\sigma I)_3} \tag{12.9}$$

and a certain linear combination of

$$m_{(j,k)}\left(\frac{n}{2}-j\right)+m_{(k,j)}\left(\frac{n}{2}-k\right), \tag{12.10}$$

$$m_{(i,j)}\left(\frac{n}{2}-i\right)+m_{(j,i)}\left(\frac{n}{2}-j\right) \text{ and } m_{(i,k)}\left(\frac{n}{2}-i\right)+m_{(k,i)}\left(\frac{n}{2}-k\right).$$

Now the relation (12.8) yields a cancellation of the six terms in (12.9) against the six terms in (12.10).

We finish with some comments on the generating function $\mathcal{G}$ (see (9.11)). The relation

$$\mathcal{L}_{2N} = -(-1)^N \Lambda_{2N}/(N-1)!, \; N \geq 1 \tag{12.11}$$

between Λ_{2N} and the leading coefficient of the polynomials $Q_{2N}^{res}(\lambda)$ (see (8.10)) implies $\mathcal{G}(r) = \mathcal{L}(r)$, where

$$\mathcal{L}(r) \stackrel{\text{def}}{=} -\sum_{N\geq 0} \mathcal{L}_{2N} \frac{r^N}{N!}, \quad \mathcal{L}_0 \stackrel{\text{def}}{=} -1.$$

In these terms, Conjecture 9.2 reads

$$\mathcal{L}(r^2/4) = \sqrt{v(r)}. \tag{12.12}$$

Thus, for the proof of Conjecture 9.2 it suffices to prove the relation (12.11) and to establish (12.12). (12.11) should be a consequence of the recursive structure of the Q-polynomials (see the discussion in Section 8). For a proof of (12.12) see [J2] (Proposition 4.2).

13. Appendix

Here we present self-contained proofs of the conformal covariance of P_4 and P_6 in the respective critical dimensions and for general metrics. These proofs serve as illustrations of the general lines of arguments of this paper.

13.1. **The critical Paneitz operator.** In the present section we prove the conformal covariance of the critical Paneitz operator. We also discuss an extension of the argument to general dimensions.

Let $n = 4$. We write P_4 (see (1.2)) in the form

$$P_4 = (P_2^2)^0 - 4\delta(\mathsf{P}\#d) = \mathcal{P}_4^0 - 4\mathcal{T}_1. \tag{13.1}$$

The following result is a special case of Theorem 3.1.

Lemma 13.1. *For $n = 4$ and $\mathcal{P}_4 = P_2^2$,*

$$(d/dt)|_0 \left(e^{4t\varphi}\mathcal{P}_4(e^{2t\varphi}g)\right) = [P_2(g), [P_2(g), \varphi]].$$

Proof. The relation

$$e^{3\varphi}P_2(e^{2\varphi}g) = P_2(g)e^{\varphi}$$

implies

$$e^{4\varphi}\hat{P}_2^2 = e^{\varphi}P_2e^{-2\varphi}P_2e^{\varphi},$$

where P_2 and $\hat{P}_2$ are the respective Yamabe operators for g and $\hat{g} = e^{2\varphi}g$. Hence

$$(d/dt)|_0(e^{4t\varphi}\mathcal{P}_4(e^{2t\varphi}g)) = \varphi P_2^2 - 2P_2\varphi P_2 + P_2^2\varphi$$

$$= [P_2, [P_2, \varphi]].$$

The proof is complete. □

Lemma 13.1 implies that

$$(d/dt)|_0 \left(e^{4t\varphi}\mathcal{P}_4^0(e^{2t\varphi}g)\right) = [P_2, [P_2, \varphi]]^0 = [\Delta, [\Delta, \varphi]]^0.$$

But $[\Delta, \varphi]u = 2(d\varphi, du) + u\Delta\varphi$ gives

$$\begin{aligned} [\Delta, [\Delta, \varphi]]^0 u &= 2\Delta(du, d\varphi) + 2(d\Delta\varphi, du) - 2(d\varphi, d\Delta u) \\ &= 4(\mathrm{Hess}(u), \mathrm{Hess}(\varphi)) + 4(d\Delta\varphi, du) + 4(\mathrm{Ric}, du \otimes d\varphi) \end{aligned} \tag{13.2}$$

using Weitzenböck's formula. Here $\mathrm{Hess}(X,Y)(u) = \langle \nabla_X(du), Y\rangle$ is the covariant Hessian of u. In order to determine the conformal variation of $\mathcal{T}_1 = \delta(\mathsf{P}\#d)$, we write $\mathcal{T}_1(u) = -(d\mathsf{J}, du) - (\mathsf{P}, \mathrm{Hess}(u))$. The transformation laws

$$e^{2\varphi}\hat{\mathsf{J}} = \mathsf{J} - \Delta\varphi - \left(\frac{n}{2}-1\right)|d\varphi|^2,$$

$$\hat{\mathsf{P}} = \mathsf{P} - \mathrm{Hess}(\varphi) - \frac{1}{2}|d\varphi|^2 g + d\varphi \otimes d\varphi \tag{13.3}$$

and

$$\widehat{\mathrm{Hess}}(u) = \mathrm{Hess}(u) - du \otimes d\varphi - d\varphi \otimes du + (du, d\varphi)g \tag{13.4}$$

imply

$$\begin{aligned} (d/dt)|_0 \left(e^{4t\varphi}\mathcal{T}_1(e^{2t\varphi}g)u\right) &= (d\Delta\varphi, du) + \mathsf{J}(du, d\varphi) \\ &\quad + (\mathrm{Hess}(u), \mathrm{Hess}(\varphi)) + 2(\mathsf{P}, du \otimes \varphi). \end{aligned} \tag{13.5}$$

Now combining (13.2) and (13.5) with $\mathrm{Ric} = 2\mathsf{P} + \mathsf{J}g$, yields

$$(d/dt)|_0 \left(e^{4t\varphi}P_4(e^{2t\varphi}g)u\right) = 0.$$

This proves the infinitesimal conformal covariance of P_4.

The above calculations also confirm that

$$4(d/dt)|_0 \left(e^{4t\varphi}\mathcal{T}_1(e^{2t\varphi}g)\right) = [\mathcal{T}_0(g), [\mathcal{T}_0(g), \varphi]]^0. \tag{13.6}$$

This is a special case of Theorem 3.2.

In the above argumentation, one can replace the calculation of first order terms by the following reasoning. P_4 and $\mathcal{T}_1$ both are self-adjoint without constant term. It follows that the second-order differential operators on the right-hand sides of (13.2) and (13.5) are self-adjoint without constant terms. Two such operators coincide iff their main parts coincide. Hence it suffices to compare the coefficients of $\mathrm{Hess}(u)$. In particular, this argument avoids to invoke Weitzenböck's formula.

Finally, we discuss how the above arguments extend to general dimensions $n \geq 3$. In that case, the recursive formula for P_4 reads

$$P_4 = \mathcal{P}_4^0 - 4\mathcal{T}_1 + \left(\frac{n}{2}-2\right) Q_4. \tag{13.7}$$

This formula follows from (1.2) by direct calculation.

We use (13.7) to prove the conformal covariance of P_4. First of all, analogous calculations as above show that

$$(d/dt)|_0 \left(e^{(\frac{n}{2}+2)t\varphi}\mathcal{P}_4(e^{2t\varphi}g)e^{-(\frac{n}{2}-2)t\varphi}\right) = [P_2(g),[P_2(g),\varphi]].$$

This result implies

$$(d/dt)|_0 \left[e^{(\frac{n}{2}+2)t\varphi}\mathcal{P}_4^0(e^{2t\varphi}g)(e^{-(\frac{n}{2}-2)t\varphi}u)\right] + \left(\frac{n}{2}-2\right)\mathcal{P}_4^0(g)(\varphi)u = [P_2(g),[P_2(g),\varphi]]^0 u.$$

Hence

$$\begin{aligned}(d/dt)|_0 &\left[e^{(\frac{n}{2}+2)t\varphi}\left(\mathcal{P}_4^0 + \left(\frac{n}{2}-2\right)Q_4\right)(e^{2t\varphi}g)(e^{-(\frac{n}{2}-2)t\varphi}u)\right] \\ &= -\left(\frac{n}{2}-2\right)\mathcal{P}_4^0(g)(\varphi)u + \left(\frac{n}{2}-2\right)(\mathcal{P}_4^0(g)-4\mathcal{T}_1(g))(\varphi)u + [P_2(g),[P_2(g),\varphi]]^0 u \\ &= [P_2(g),[P_2(g),\varphi]]^0 u - \left(\frac{n}{2}-2\right)4\mathcal{T}_1(g)(\varphi)u.\end{aligned}$$

Here we have used the variational formula

$$(d/dt)|_0(e^{4t\varphi}Q_4(e^{2t\varphi}g)) = (\mathcal{P}_4^0(g) - 4\mathcal{T}_1(g))(\varphi) \tag{13.8}$$

for Q_4. For the proof of (13.8) it is natural to combine the formula

$$Q_4 = -P_2(Q_2) - Q_2^2 + 16v_4 \tag{13.9}$$

with the conformal transformation laws for P_2, Q_2 and P. The latter formula immediately shows that the left hand side of (13.8) is of the form

$$P_2^2 - 4\delta(\mathsf{P}\#d) + LOT = \mathcal{P}_4 - 4\mathcal{T}_1 + LOT.$$

The actual calculation yields (13.8).

Next, (13.6) generalizes to

$$4(d/dt)|_0 \left(e^{(\frac{n}{2}+2)t\varphi}\mathcal{T}_1(e^{2t\varphi}g)e^{-(\frac{n}{2}-2)t\varphi}\right) = [\mathcal{T}_0(g),[\mathcal{T}_0(g),\varphi]]^0 - \left(\frac{n}{2}-2\right)4\mathcal{T}_1(g)(\varphi).$$

Combining these results yields the conformal covariance of P_4.

We finish with two comments. In contrast to the critical case, the proof for $n \ne 4$ requires conformal variation of Q_4. Only by recognizing the right-hand side of (13.9) as the fourth order Q-curvature leads to the identification of the right-hand side of (13.7) as the Paneitz operator.

13.2. **The critical P_6 for general metrics.** Here we prove the infinitesimal conformal covariance of the operator

$$\mathbf{P}_6 \stackrel{\text{def}}{=} \left[2(P_2P_4 + P_4P_2) - 3P_2^3\right]^0 - 48\delta(\mathsf{P}^2\#d) - 8\delta(\mathcal{B}\#d) \tag{13.10}$$

in dimension $n = 6$ (see (4.4)). The result implies the conformal covariance

$$e^{6\varphi}\mathbf{P}_6(e^{2\varphi}g) = \mathbf{P}_6(g).$$

Lemma 13.2. *Let $n = 6$ and $\mathcal{P}_6 = 2(P_2P_4 + P_4P_2) - 3P_2^3$. Then*

$$(d/dt)|_0\left(e^{6t\varphi}\mathcal{P}_6(e^{2t\varphi}g)\right) = 4[\mathcal{M}_4(g),[P_2(g),\varphi]] + 2[P_2(g),[\mathcal{M}_4(g),\varphi]]. \tag{13.11}$$

Proof. The relations

$$e^{4\varphi} P_2(e^{2\varphi} g) = P_2(g) e^{2\varphi} \quad \text{and} \quad e^{5\varphi} P_4(e^{2\varphi} g) = P_4(g) e^{\varphi}$$

imply

$$\hat{P}_2 \hat{P}_4 = e^{-4\varphi} P_2 e^{-3\varphi} P_4 e^{\varphi}, \quad \hat{P}_4 \hat{P}_2 = e^{-5\varphi} P_4 e^{-3\varphi} P_2 e^{2\varphi}$$

and

$$\hat{P}_2^3 = e^{-4\varphi} P_2 e^{-2\varphi} P_2 e^{-2\varphi} P_2 e^{2\varphi}.$$

Hence

$$\begin{aligned}(d/dt)|_0 \left(\mathcal{P}_6(e^{2t\varphi} g)\right) &= 2\left(-4\varphi P_2 P_4 - 3 P_2 \varphi P_4 + P_2 P_4 \varphi\right) \\ &+ 2\left(-5\varphi P_4 P_2 - 3 P_4 \varphi P_2 + 2 P_4 P_2 \varphi\right) \\ &- 3\left(-4\varphi P_2^3 - 2 P_2 \varphi P_2^2 - 2 P_2^2 \varphi P_2 + 2 P_2^3 \varphi\right).\end{aligned}$$

The latter formula is equivalent to

$$\begin{aligned}(d/dt)|_0 \left(e^{6t\varphi} \mathcal{P}_6(e^{2t\varphi} g)\right) &= 2\left(-2[P_2, \varphi] P_4 + P_2 [P_4, \varphi]\right) \\ &+ 2\left(-[P_4, \varphi] P_2 + 2 P_4 [P_2, \varphi]\right) \\ &- 3\left(-2[P_2, \varphi] P_2^2 + 2 P_2^2 [P_2, \varphi]\right) \\ &= 4\left[P_4, [P_2, \varphi]\right] + 2\left[P_2, [P_4, \varphi]\right] - 6\left[P_2^2, [P_2, \varphi]\right].\end{aligned}$$

Now $[P_2^2, [P_2, \varphi]] = [P_2, [P_2^2, \varphi]]$ yields the assertion. $\square$

A similar calculation shows that in general dimensions,

$$(d/dt)|_0 \left(e^{(\frac{n}{2}+3)t\varphi} \mathcal{P}_6(e^{2t\varphi} g) e^{-(\frac{n}{2}-3)t\varphi}\right) = 4[\mathcal{M}_4, [P_2, \varphi]] + 2[P_2, [\mathcal{M}_4, \varphi]].$$

Lemma 13.2 implies that

$$(d/dt)|_0 \left(e^{6t\varphi} \mathcal{P}_6^0(e^{2t\varphi} g)\right) = 4[\mathcal{M}_4, [P_2, \varphi]]^0 + 2[P_2, [\mathcal{M}_4, \varphi]]^0. \tag{13.12}$$

The right-hand side coincides with

$$4[\mathcal{M}_4^0, [\mathcal{M}_2^0, \varphi]]^0 + 2[\mathcal{M}_2^0, [\mathcal{M}_4^0, \varphi]]^0.$$

Using

$$\mathcal{M}_4^0 = -4\delta(\mathsf{P}\#d) = -4\mathcal{T}_1 \quad \text{and} \quad \mathcal{M}_2^0 = \Delta$$

this sum equals

$$-16[\mathcal{T}_1, [\Delta, \varphi]]^0 - 8[\Delta, [\mathcal{T}_1, \varphi]]^0. \tag{13.13}$$

As a self-adjoint second-order operator which annihilates constants, this operator is determined by its main part. A calculation shows that

$$[\mathcal{T}_1, [\Delta, \varphi]] u = -2(\mathsf{P}, \mathrm{Hess}(d\varphi, du)) + 2(d\varphi, d(\mathsf{P}, \mathrm{Hess}(u))) + \text{first order terms}.$$

The main part of this operator is given by

$$-4\mathsf{P}_{ij} \,\mathrm{Hess}_k^i(\varphi) \,\mathrm{Hess}^{jk}(u) + 2\nabla_i(\mathsf{P})_{jk} \,\mathrm{Hess}^{jk}(u) \varphi^i.$$

Similarly, for the second term in (13.13) we find

$$[\Delta, [\mathcal{T}_1, \varphi]] u = -2\Delta(\mathsf{P}, d\varphi \otimes du) + 2(\mathsf{P}, d\varphi \otimes d\Delta u) + \text{first order terms}.$$

The main part of this operator is given by

$$-4\nabla_k(\mathsf{P})_{ij}\varphi^i \operatorname{Hess}^{jk}(u) - 4\mathsf{P}_{ij} \operatorname{Hess}^{ik}(\varphi) \operatorname{Hess}_k^j(u).$$

Thus, for the main part of (13.13) we find the formula

$$\begin{aligned} 96\mathsf{P}_{ij} \operatorname{Hess}^{ik}(\varphi) \operatorname{Hess}_k^j(u) - 32\nabla_i(\mathsf{P})_{jk} \operatorname{Hess}^{jk}(u)\varphi^i + 32\nabla_k(\mathsf{P})_{ij} \operatorname{Hess}^{jk}(u)\varphi^i \\ = 96\mathsf{P}_{ij} \operatorname{Hess}^{ik}(\varphi) \operatorname{Hess}_k^j(u) + 32\mathcal{C}_{kij} \operatorname{Hess}^{jk}(u)\varphi^i. \end{aligned} \tag{13.14}$$

Next, the main part of $(d/dt)|_0\,(e^{6t\varphi}\mathcal{T}_2(e^{2t\varphi}g))$ is

$$\operatorname{Hess}_{ik}(\varphi)\mathsf{P}_j^k \operatorname{Hess}^{ij}(u) + \mathsf{P}_{ik} \operatorname{Hess}_j^k(\varphi) \operatorname{Hess}^{ij}(u) = 2\operatorname{Hess}_{ik}(\varphi)\mathsf{P}_j^k \operatorname{Hess}^{ij}(u). \tag{13.15}$$

Finally, by the transformation law

$$e^{2\varphi}\hat{\mathcal{B}}_{ij} = \mathcal{B}_{ij} - (n-4)(\mathcal{C}_{ikj} + \mathcal{C}_{jki})\varphi^k + (n-4)\mathsf{C}_{kijl}\varphi^k\varphi^l \tag{13.16}$$

for the Bach tensor, the main part of the conformal variation of $\delta(\mathcal{B}\#du)$ is

$$4\mathcal{C}_{ikj}\varphi^k \operatorname{Hess}^{ij}(u). \tag{13.17}$$

Now

$$(13.14) - 48 \times (13.15) - 8 \times (13.17) = 0$$

implies the infinitesimal conformal covariance of the operator (13.10). □

The proof shows that the only property of the Bach tensor $\mathcal{B}_{ij}$ which enters is that its conformal variation is given by a multiple of

$$(\mathcal{C}_{ikj} + \mathcal{C}_{jki})\varphi^k.$$

In the proof of Theorem 11.1, a similar property of $\Omega^{(2)}$ plays an analogous role (with the symmetrized Cotton tensor replaced by the higher Cotton tensor $\mathcal{C}^{(2)}$).

The above calculations directly confirm the following special case of Theorem 3.2. We recall that $\mathcal{T}_0 = -\Delta$.

Corollary 13.1. *In the locally conformally flat case,*

$$6(d/dt)|_0\left(e^{6t\varphi}\mathcal{T}_2(e^{2t\varphi}g)\right) = [\mathcal{T}_0,[\mathcal{T}_1,\varphi]]^0 + 2[\mathcal{T}_1,[\mathcal{T}_0,\varphi]]^0.$$

Proof. This is an identity of self-adjoint second-order operators which annihilate constants. It suffices to compare the main parts of both sides. The result follows from the above calculations. □

For any constant α, the operator $P_4' = P_4 + \alpha|\mathsf{C}|^2$ is conformally covariant. The same proof as above shows that in dimension $n = 6$,

$$P_6' = (2(P_2P_4' + P_4'P_2) - 3P_2^3)^0 - 48\delta(\mathsf{P}^2\#d) - 8\delta(\mathcal{B}\#d)$$

is conformally covariant. The latter conformally covariant cube of the Laplacian differs from the GJMS-operator P_6 by a multiple of the self-adjoint operator $\delta(|\mathsf{C}|^2du)$. But note that the latter operator is not conformally covariant in dimensions $n \neq 6$. In other words, universality is lost.

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